Homework 12

(1) Compute
$$\int_0^\infty \frac{x^2}{x^4+x^2+1}\,dx.$$

(2) Compute
$$\int_{-\infty}^\infty \frac{\cos \pi x}{x^2-2x+2}\,dx$$
by integrating $f(z) = \frac{e^{\pi i z}}{z^2-2z+2}$ over a semi-circular path.

(3) Find, without using a factorization, an $r > 0$ such that the polynomial $p(z) = z^3 - 4z^2 + z - 4$ has exactly two roots inside the circle $|z| < r$.

(4) Prove that for any $R > 0$ there exists N such that
$$p_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}$$
has no zeros inside $|z| < R$ for all $n \geq N$. (Hint: Use that $p_n \to e^z$ uniformly on $D(0; R)$, so that $|p_n(z) - e^z| < |e^z|$ on $|z| = R$ for n large and then Rouché's theorem applies.)