Homework 12, Due April 30

1. Let G be a region (i.e. an open connected set). Let f be holomorphic on G such that $|f|$ is constant on G. Show that f is constant on G. (Note: this fact is used in the proof of the maximum modulus principle)
2. Let G be a region and let $\langle f_n \rangle$ be a sequence of holomorphic functions which converges to a function f uniformly on compact subsets of G. Prove that f is holomorphic on G.
3. (Schwarz's lemma) Let f be a holomorphic function on $D(0;1)$ with $|f(z)| \leq 1$ for all $|z| < 1$ and $f(0) = 0$.
   a. Define $f_1(z) = \frac{f(z)}{z}$ for $z \neq 0$ in $D(0;1)$. Prove that $z = 0$ is a removable singularity of f_1.
   b. Prove that $|f_1(z)| \leq \frac{1}{r}$ on $D(0;r)$ for all $0 < r < 1$. (Hint: use the maximum modulus principle.)
   c. Conclude that $|f(z)| \leq |z|$ for all $z \in D(0;1)$ and if equality holds for some $z \neq 0$, then $f(z) = cz$ for some c with $|c| = 1$.
4. Let $f(z) = \frac{1}{z^4+1}$.
   a. Find the solutions $\{z_1, z_2, z_3, z_4\}$ of $z^4 = -1$ (hint: write $z = re^{i\theta}$ and solve the equation for the exponential) and compute the residues at the two roots in the upper half plane by using the formula
   $$\text{Res }(f, z_k) = \lim_{z \to z_k} (z - z_k) f(z).$$
   b. Let γ_R be defined by $\gamma_R(t) = Re^{it}$ with $0 \leq t \leq \pi$. Prove that
   $$\lim_{R \to \infty} \left| \int_{\gamma_R} f(z)\, dz \right| = 0.$$
   c. Let Γ_R be the join of γ_R and $\tilde{\gamma}_R$, where $\tilde{\gamma}_R$ is the directed line segment $[-R, R]$. Compute $\int_{\Gamma_R} f(z)\, dz$ using the formula
   $$\int_{\Gamma_R} f(z)\, dz = 2\pi i \left(\text{Res }(f, z_1) + \text{Res }(f, z_2)\right),$$
   where z_1, z_2 are the poles inside Γ_R for $R > 1$.
   d. Use (b) and (c) to show that
   $$\lim_{R \to \infty} \int_{-R}^{R} \frac{1}{x^4+1}\, dx = \frac{\pi\sqrt{2}}{2}.$$