Homework 12

(1) If G is a region and $f, g \in H(G)$ are such that $f(z)g(z) = 0$ for all $z \in G$, then either $f(z) = 0$ for all $z \in G$ or $g(z) = 0$ for all $z \in G$.

(2) (Quals '02) Let $f, g : \{z : |z| < 1\} \to \mathbb{C}$ be holomorphic functions such that $|f(z)| = |g(z)|$ for all $|z| < 1$. Prove that every zero of g is also a zero of f of the same multiplicity and that thus $f = \lambda g$ for some λ with modulus one.

(3) (Schwarz's lemma) Let f be a holomorphic function on $D(0;1)$ with $|f(z)| \leq 1$ for all $|z| < 1$ and $f(0) = 0$.

   a. Define $f_1(z) = \frac{f(z)}{z}$ for $z \neq 0$ in $D(0;1)$. Prove that $z = 0$ is a removable singularity of f_1.

   b. Prove that $|f_1(z)| \leq \frac{1}{r}$ on $D(0;r)$ for all $0 < r < 1$. (Hint: use the maximum modulus principle.)

   c. Conclude that $|f(z)| \leq |z|$ for all $z \in D(0;1)$. Moreover if equality holds for some $z_0 \neq 0$, then there exists c with $|c| = 1$ such that $f(z) = cz$ for all $z \in D(0;1)$.