Homework 12.

(1) Let $G \subset \mathbb{C}$ be open and let f be holomorphic on G. Let $G^* = \{z : \bar{z} \in G\}$ and define $f^*(z) = \overline{f(\bar{z})}$ for all $z \in G^*$. Prove that f^* is holomorphic on G^* and express $f^*(z)'$ in terms of f'.

(2) (Quals '06) Let f be a holomorphic function on $|z| < 1$. Assume $f(\frac{1}{n}) \in \mathbb{R}$ for $n \geq 2$. Prove $f(x) \in \mathbb{R}$ for all $-1 < x < 1$. (Hint: Use the previous problem.)

(3) Compute

$$\int_0^\infty \frac{x^2}{x^4 + x^2 + 1}\, dx.$$

by integrating $f(z) = \frac{z^2}{z^4+z^2+1}$ over a semi-circular path and using the residue theorem.

(4) Compute

$$\int_{-\infty}^\infty \frac{\cos \pi x}{x^2 - 2x + 2}\, dx$$

by integrating $f(z) = \frac{e^{\pi i z}}{z^2-2z+2}$ over a semi-circular path.