Homework 11

(1) (Quals '04) Let $G \subset \mathbb{C}$ be a region (i.e. an open and connected set) and let $f : G \to \mathbb{C}$ be a holomorphic function such that $|f(z)| = C$ for all $z \in G$. Prove that f is constant on G.
(2) (Quals '95) Let f be an entire function on $\mathbb{C}$ and assume that $|f(z)| \leq A|z|^k + B$ for some constants A, B, integer k and all $z \in \mathbb{C}$. Prove that f is a polynomial.
(3) (Quals '98) Let $G \subset \mathbb{C}$ be an open set containing the closed disk $\overline{D(a;r)} = \{z : |z - a| \leq r\}$. Let $\langle f_n \rangle$ be a sequence of holomorphic functions on G such that $f_n(z) \to 0$ uniformly on $\{z : |z - a| = r\}$. Prove that $f_n(z) \to 0$ for all z in the open disk $D(a;r)$.
(4) (Quals '95) Let $G \subset \mathbb{C}$ be a region and let $\langle f_n \rangle$ be a sequence of holomorphic functions on G, which converges uniformly on every compact subset of G to a function f. Prove that f is holomorphic on G.