Homework 11

(1) (Quals 1995) Let f be an entire function on $\mathbb{C}$ and assume that $|f(z)| \leq A|z|^k + B$ for some constants A, B, integer k and all $z \in \mathbb{C}$. Prove that f is a polynomial.

(2) (Quals 1995) Let $G \subset \mathbb{C}$ be a connected open set and let $\langle f_n \rangle$ be a sequence of holomorphic functions on G, which converges uniformly on every compact subset of G to a function f. Prove that f is holomorphic on G.

(3) (Quals 2000) Let f be an entire function such that $|f(z)| \leq Me^{\mathrm{Re}\ z}$ for some $M > 0$ and all $z \in \mathbb{C}$. Prove that there exists $C \in \mathbb{C}$ such that $f(z) = Ce^z$.

(4) Let G be open and connected and f, g analytic on G such that $f(z)g(z) = 0$ for all $z \in G$. Prove that either $f(z) = 0$ for all $z \in G$ or $g(z) = 0$ for all $z \in G$.

(5) (Quals '02) Let $f, g : \{z : |z| < 1\} \to \mathbb{C}$ be analytic functions such that $|f(z)| = |g(z)|$ for all $|z| < 1$. Prove that every zero of g is also a zero of f of the same multiplicity and that thus $f = \lambda g$ for some λ with modulus one.