Homework 11.

(1) Let $F : [0,1] \to \mathbb{R}$ such that $F'(x)$ exists a.e. and satisfies $F' \in L^1([0,1])$. Assume F is continuous at 0 and absolutely continuous on $[\epsilon, 1]$ for all $\epsilon > 0$. Prove that F is absolutely continuous on $[0,1]$ and thus of bounded variation on $[0,1]$.

(2) Let $a > b > 0$ and define $F(0) = 0$, $F(x) = x^a \sin \frac{1}{x^b}$ for $0 < x \leq 1$. Prove that F is of bounded variation on $[0,1]$ (Hint: Use Problem 2 to prove that F is absolutely continuous).

(3) (Quals January 2007)Let $f : [0,1] \to \mathbb{R}$. Prove that the following are equivalent.
   a. f is absolutely continuous, $f'(x) \in \{0,1\}$ a.e., and $f(0) = 0$.
   b. There exists a measurable set $A \subset [0,1]$ such that $f(x) = m\,(A \cap (0,x))$.

(4) (Quals January 2004)Let f_n be absolutely continuous on $[0,1]$ and let $f_n(0) = 0$. Assume that
$$\int_0^1 |f_n'(x) - f_m'(x)|\, dx \to 0$$
as $m, n \to \infty$. Prove that f_n converges uniformly to a function f on $[0,1]$ and that f is absolutely continuous on $[0,1]$.

(5) (Quals August 2002) Let $f : [a,b] \to [c,d]$ be an increasing absolutely continuous function and let $g : [c,d] \to \mathbb{R}$ be an absolutely continuous function. Prove that the composition $g \circ f : [a,b] \to \mathbb{R}$ is absolutely continuous.