Homework 11.

(1) Let G be open and connected and f, g analytic on G such that $f(z)g(z) = 0$ for all $z \in G$. Prove that either $f(z) = 0$ for all $z \in G$ or $g(z) = 0$ for all $z \in G$.
(2) (Quals '02) Let $f, g : \{z : |z| < 1\} \to \mathbb{C}$ be analytic functions such that $|f(z)| = |g(z)|$ for all $|z| < 1$. Prove that every zero of g is also a zero of f of the same multiplicity and that thus $f = \lambda g$ for some λ with modulus one.
(3) (Quals '06) Let f be a holomorphic function on $|z| < 1$.
    a. Let $g(z) = \overline{f(\overline{z})}$ on $|z| < 1$. Prove g is holomorphic on $|z| < 1$.
    b. Assume $f(\frac{1}{n}) \in \mathbb{R}$ for $n \geq 2$. Prove $f(x) \in \mathbb{R}$ for all $-1 < x < 1$.