Homework 10, Additional Problems.

(1) **a.** Let $F : [a, b] \to \mathbb{R}$ be a function of bounded variation and let $a < c < b$, Prove that $T_a^c(F) + T_c^b(F) = T_a^b(F)$. (Here $T_a^b(F)$ denotes the total variation of F on $[a, b]$). Deduce that the function $x \mapsto T_a^x(F)$ is increasing.

b. Let F be as in part **a.** Prove that

$$\int_a^b |F'(x)|\, dx \le T_a^b(F).$$

(Hint: Use $|F(x+h) - F(x)| \le T_x^{x+h}(F)$ to get that $|F'(x)| \le (T_a^x(F))'$ a.e)

(2) Let $a \le b$ and define $F(0) = 0$, $F(x) = x^a \sin \frac{1}{x^b}$ for $0 < x \le 1$. Prove that F is not of bounded variation on $[0, 1]$.