Homework 10

(1) Evaluate (without parametrizing, but using Cauchy's Integral Theorem) $\int_\gamma \frac{1}{1+z^2}\,dz$ for

- **a.** $\gamma(t) = 1 + e^{it}$ $(0 \le t \le 2\pi)$.
- **b.** $\gamma(t) = -i + e^{it}$ $(0 \le t \le 2\pi)$.
- **c.** $\gamma(t) = 2e^{it}$ $(0 \le t \le 2\pi)$.
- **d.** $\gamma(t) = 3i + 3e^{it}$ $(0 \le t \le 2\pi)$.

(2) Let $\alpha \in \mathbb{C}$ with $|\alpha| \neq 1$. Compute

$$\int_0^{2\pi} \frac{dt}{1 - 2\alpha\cos t + \alpha^2}$$

by computing (show how they are related)

$$\frac{i}{\alpha}\int_\gamma \frac{1}{(z-\alpha)(z-\frac{1}{\alpha})}\,dz,$$

where $\gamma(t) = e^{it}$ with $0 \le t \le 2\pi$.

(3) (Quals 1999) Prove that

$$\int_0^{\pi} e^{\cos\theta}\cos(\sin\theta)\,d\theta = \pi.$$

(Hint: Consider $\int_\gamma \frac{e^z}{z}\,dz$, where $\gamma(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$.)

(4) Compute

$$\int_0^{2\pi} \frac{\cos t}{5 - 4\cos t}\,dt.$$

(5) Which of the following sets are starlike? In case they are starlike, find a star center of the set.

- **a.** $\{z \in \mathbb{C} : |z| < 1 \text{ and } |z+1| > \sqrt{2}\}$,
- **b.** $\{z \in \mathbb{C} : |z| < 1 \text{ and } |z-2| > \sqrt{5}\}$,

  (Hint: Both sets are "sickle shaped". To find possible star centers, find the point of intersection of the tangent lines to the boundary curve closest to the origin at the points of intersection of the boundary curves.)

(6) (Quals 1998) Let $G \subset \mathbb{C}$ be an open set containing the closed disk $\overline{B(a;r)} = \{z : |z-a| \le r\}$. Let $\langle f_n \rangle$ be a sequence of holomorphic functions on G such that $f_n(z) \to 0$ uniformly on $\{z : |z-a| = r\}$. Prove that $f_n(z) \to 0$ for all z in the open disk $B(a;r)$. (Hint: Use Cauchy's Integral Formula.)