Homework 10, Additional Problems.

(1) **a.** Let $F : [a, b] \to \mathbb{R}$ be a function of bounded variation and let $a < c < b$, Prove that $T_a^c(F) + T_c^b(F) = T_a^b(F)$. (Here $T_a^b(F)$ denotes the total variation of F on $[a, b]$).

b. Let F be as in part **a.** Prove that

$$\int_a^b |F'(x)|\, dx \leq T_a^b(F).$$

(Hint: Use $|F(x + h) - F(x)| \leq T_x^{x+h}(F)$ to get that $|F'(x)| \leq (T_a^x(F))'$ a.e)

(2) Let $a \leq b$ and define $F(0) = 0$, $F(x) = x^a \sin \frac{1}{x^b}$ for $0 < x \leq 1$. Prove that F is not of bounded variation on $[0, 1]$.

(3) Hint for Problem 15: It suffices to prove that the positive and negative variation functions $P_a^x F$ and $N_a^x F$ are continuous. For this indicate that it suffices to prove that the total variation function $T_a^x F$ is continuous at say $\bar{x}$ if F is continuous at $\bar{x} \in [a, b]$. Now approximate $T_a^b F$ using a partition within, say, $\frac{\epsilon}{3}$. By refining we can assume $\bar{x} = x_k$ for some $1 \leq k < n$ (where $x_0 = a < \cdots < x_n = b$ is the partition used to approximate $T_a^b F$). Now use continuity at $\bar{x}$ and part a) of Problem 1 above to estimate $T_{x_{k-1}}^{x_{k+1}} F$ by $\frac{\epsilon}{3} + |f(x_k) - f(x_{k-1})| + |f(x_{k+1}) - f(x_k)|$.