Homework 10.

(1) (Quals '95) Let f be an entire function on $\mathbb{C}$ and assume that $|f(z)| \leq A|z|^k + B$ for some constants A, B, natural number k and all $z \in \mathbb{C}$. Prove that f is a polynomial. (Hint: Prove that the derivative $f^{(k+1)}(z) = 0$ for all z, using the Cauchy integral formula for $f^{(k+1)}$.)

(2) (Quals '98) Let $G \subset \mathbb{C}$ be an open set containing the closed disk $\overline{B(a;r)} = \{z : |z-a| \leq r\}$. Let $\langle f_n \rangle$ be a sequence of analytic functions on G such that $f_n(z) \to 0$ uniformly on $\{z : |z-a| = r\}$. Prove that $f_n(z) \to 0$ for all z in the open disk $B(a;r)$.

(3) (Quals '95) Let $G \subset \mathbb{C}$ be a region and let $\langle f_n \rangle$ be a sequence of analytic functions on G, which converges uniformly on every compact subset of G to a function f. Prove that f is analytic on G.

(4) (Quals '02) Let $\Omega \subset \mathbb{C}$ be an open set containing the unit disk $\{z : |z| \leq 1\}$ and let $f : \Omega \to \mathbb{C}$ be a analytic function such that $|f(z)| > |f(0)|$ for all $|z| = 1$. Prove that f has a zero in $|z| < 1$.