# A SIMPLE COMPLEX ANALYSIS AND AN ADVANCED CALCULUS PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

ANTON R. SCHEP

It is hard not to have Ray Redheffer's title of [2] as a reaction to another article on the Fundamental Theorem of Algebra. In fact at least 28 notes have appeared in this Monthly about this theorem. In this note we present nevertheles two proofs of the Fundamental Theorem of Algebra, which do not seem to have been observed before and which we think are worth recording. The first one uses Cauchy's Integral Theorem and is, in the author's opinion, as simple as the most popular complex analysis proof based on Liouville's theorem (see [3] for this and three other proofs using complex analysis). The editor of this Monthly did provide a reference to Problem 5 on pg. 126 of [1], where a proof of the Fundamental Theorem of Algebra is given based on a similar complex contour integral as here, but the details are not quite the same. The second one considers the integral obtained by parameterizing the contour integral from the first proof and uses only results from advanced calculus. This proof is similar to the proof of [4], where the same ideas were used to prove the non-emptiness of the spectrum of an element in a complex Banach algebra. There the companion matrix of a polynomial was used then to derive the Fundamental Theorem of Algebra.

**Theorem** (Fundamental Theorem of Algebra)**.** *Every polynomial of degree $n \geq 1$ with complex coefficients has a zero in $\mathbb{C}$.*

*Proof.* Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$ and assume that $p(z) \neq 0$ for all $z \in \mathbb{C}$.
First Proof: By Cauchy's integral theorem we have

$$\oint_{|z|=r} \frac{dz}{zp(z)} = \frac{2\pi i}{p(0)} \neq 0,$$

where the circle is traversed counter clockwise. On the other hand

$$\left| \oint_{|z|=r} \frac{dz}{zp(z)} \right| \leq 2\pi r \cdot \max_{|z|=r} \frac{1}{|zp(z)|} = \frac{2\pi}{\min_{|z|=r} |p(z)|} \to 0 \text{ as } r \to \infty$$

(since $|p(z)| \geq |z|^n|(1 - |a_{n-1}|/|z| - \cdots - |a_0|/|z^n|)|$), which is a contradiction.
Second Proof: Define $g : [0, \infty) \times [0, 2\pi] \to \mathbb{C}$ by $g(r, \theta) = 1/p(re^{i\theta})$. Then the function $g$ is continuous on $[0, \infty) \times [0, 2\pi]$ and has continuous partials on $(0, \infty) \times (0, 2\pi)$, satisfying $\frac{\partial g}{\partial \theta} = ir \cdot \frac{\partial g}{\partial r}$.

Define now $F : [0, \infty) \to \mathbb{C}$ by $F(r) = \int_0^{2\pi} g(r, \theta)\, d\theta$. Then by Leibniz's rule for differentiation under the integral sign we have for all $r > 0$

$$irF'(r) = ir \int_0^{2\pi} \frac{\partial g}{\partial r}\, d\theta = \int_0^{2\pi} \frac{\partial g}{\partial \theta}\, d\theta = g(r, 2\pi) - g(r, 0) = 0.$$

Hence $F'(r) = 0$ for all $r > 0$. This implies that $F$ is constant on $[0, \infty)$ with $F(r) = F(0) = \frac{2\pi}{p(0)} \neq 0$. On the other hand $|p(z)| \to \infty$ uniformly as $|z| \to \infty$ implies that $g(r, \theta) \to 0$ as $r \to \infty$ uniformly in $\theta$. Therefore $F(r) \to 0$ as $r \to \infty$, which is a contradiction. $\square$

## References

[1] N. Levinson and R. M. Redheffer, *Complex variables*, Holden-Day Inc., San Francisco, Calif., 1970.

[2] R. M. Redheffer, What! Another Note Just on the Fundamental Theorem of Algebra? (in Classroom Notes), this Monthly **71** (1964) 180 – 185.

[3] R. Remmert, *Theory of complex functions*, Graduate Texts in Mathematics, vol. 122, Springer-Verlag, New York, 1991, Translated from the second German edition by Robert B. Burckel, Readings in Mathematics.

[4] D. Singh, The spectrum in a Banach algebra, this Monthly **113** (2006) 756 – 758.

Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA
*E-mail address*: `schep@math.sc.edu`