# LOZANOVSKIĬ'S PROOF OF DUNFORD'S THEOREM

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Let $(X, \mu)$ and $(Y, \nu)$ be $\sigma$-finite measure spaces. We denote by $L^r(L^s)$ the Banach lattice $\{f(x,y) \in L^0(X \times Y, \mu \times \nu) : \| \, \|f_y\|_s \|_r < \infty\}$, where $f_y$ denotes the function $x \mapsto f(x,y)$. We denote by $\|f\|_{r,s}$ the norm $\| \, \|f_y\|_s \|_r < \infty$ of $f$. It is well known, see [B], that for $1 \leq r, s \leq \infty$ the order continuous dual of $L^r(L^s)$ equals $L^{r'}(L^{s'})$. In particular for $1 < p \leq \infty$ the space $L^\infty(L^p)$ is the dual space of $L^1(L^{p'})$, so that the unit ball of $L^\infty(L^p)$ is weak*-compact with respect to this duality. For $k \in L^\infty(L^p)$ we denote by $T_k$ the integral operator with kernel $k$, i.e.

$$T_k(f) = \int_Y k(x,y) f(y) d\nu(y) \text{ a.e.}$$

It is an exercise (by using adjoints) to show that $T_k$ is a bounded linear operator from $L^1$ into $L^p$ with $\|T_k\| = \|k\|_{\infty,p}$.

**Theorem (Dunford's theorem** [D]**).** *Let $T$ be a bounded linear operator from $L^1(Y,\nu)$ into $L^p(X,\mu)$. Then there exists $k \in L^\infty(L^p)$ such that $T = T_k$.*

*Proof (Lozanovskiĭ* [L]*).* We may assume that $T \geq 0$ and $\|T\| \leq 1$. Define $\mathcal{M} = \{U : U = (\chi_{A_1}, \ldots, \chi_{A_n}) : A_i \cap A_j = \emptyset \text{ for } i \neq j, 0 < \mu(A_i) < \infty\}$. Then for $U \in \mathcal{M}$ we denote

$$R(U) = \{k \in L^\infty(L^p) : k \geq 0 \text{ a.e. }, \|k\|_{\infty,p} \leq 1$$
$$\text{and } T_k(\chi_{A_i}) = T(\chi_{A_i}) \text{ for } i = 1, \ldots, n\}.$$

We shall first show that $R(U) \neq \emptyset$ for all $U \in \mathcal{M}$. Let $U = (\chi_{A_1}, \ldots, \chi_{A_n}) \in \mathcal{M}$. Define

$$k(x,y) = \sum_{j=1}^{n} \frac{T(\chi_{A_i})(x)\chi_{A_i}(y)}{\nu(A_i)}.$$

Now clearly $k \geq 0$, $T_k(\chi_{A_i}) = T(\chi_{A_i})$ and

$$\|k\|_{\infty,p} = \operatorname{ess\,sup}_y \left\| \sum_{i=1}^{n} \frac{T(\chi_{A_i})(x)\chi_{A_i}(y)}{\nu(A_i)} \right\|_p$$
$$\leq \operatorname{ess\,sup}_y \sum_{i=1}^{n} \chi_{A_i}(y) = 1,$$

so $k \in R(U)$. Hence $R(U) \neq \emptyset$ for all $U \in \mathcal{M}$. Next we show that $R(U)$ is weak*-closed in $L^\infty(L^p)$. Let $k_\tau \in R(U)$ such that $k_\tau \to k$ weak*. Then it easy to see that $k \geq 0$ and $\|k\| \leq 1$. Let $U = (\chi_{A_i})$ and let $g \in L^{p'}(X,\mu)$. Then $\chi_{A_i}(y)g(x) \in L^1(L^{p'})$, so $\langle k_\tau, g\chi_{A_i}\rangle \to \langle k, g\chi_{A_i}\rangle$. But by Tonelli's theorem we have

$$\langle k_\tau, g\chi_{A_i}\rangle = \int g(x)T_{k_\tau}(\chi_{A_i})(x)d\mu(x) = \int g(x)T(\chi_{A_i})(x)d\mu(x)$$

and

$$\langle k, g\chi_{A_i}\rangle = \int g(x)\left(\int k(x,y)\chi_{A_i}(y)d\nu(y)\right)d\mu(x).$$

Hence we have that

$$\int g(x)T(\chi_{A_i})(x)d\mu(x) = \int g(x)\left(\int k(x,y)\chi_{A_i}(y)d\nu(y)\right)d\mu(x)$$

for all $g \in L^{p'}$. It follows that $T(\chi_{A_i}) = T_k(\chi_{A_i})$ for all $i$. Hence $T_k \in R(U)$. We conclude from this that $R(U)$ is a weak*-compact subset of $L^\infty(L^p)$. Next we show that $\{R(U) : U \in \mathcal{M}\}$ has the finite intersection property. Let $U_1$ and $U_2 \in \mathcal{M}$. Then let $U$ denote the common refinement of $U_1$ and $U_2$, where sets of measure zero are omitted. Then $U \in \mathcal{M}$ and $R(U) \subset R(U_1) \cap R(U_2)$ by linearity of $T$ and $T_k$. Hence $\{R(U) : U \in \mathcal{M}\}$ has the finite intersection property. Weak*-compactness of the unit ball of $L^\infty(L^p)$ implies that there exists $k \in L^\infty(L^p)$ such that $k \in \cap \{R(U) : U \in \mathcal{M}\}$. This implies that $T(\chi_A) = T_k(\chi_A)$ for all measurable sets $A \subset Y$ with $0 < \nu(A) < \infty$. It follows from this that $T_k = T$ and the proof is complete.

## Generalizations and Applications

One can generalize the above proof and obtain proofs of the author's results concerning characterizations of Carleman and Hille-Tamarkin integral operators (see [S2]) and also Schachermayer's characterizations of integral operators ([S1]).

## References

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