DIFFERENTIATION OF MONOTONE FUNCTIONS

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1. Dini derivates

To define the Dini derivates (or Dini derivatives as Tao calls them) of a function we first recall the definitions of a one-sided limit superior and limit inferior. Let $f : (a, b) \to \mathbb{R}$. Then

$$\overline{\lim_{y \downarrow x}} f(y) = \inf_{\delta > 0} \sup\{f(y) : 0 < y - x < \delta\} = \lim_{\delta \downarrow 0} \sup\{f(y) : 0 < y - x < \delta\}.$$

Similarly

$$\underline{\lim_{y \downarrow x}} f(y) = \sup_{\delta > 0} \inf\{f(y) : 0 < y - x < \delta\} = \lim_{\delta \downarrow 0} \inf\{f(y) : 0 < y - x < \delta\}.$$

It is clear that $\underline{\lim}_{y \downarrow x} f(y) \le \overline{\lim}_{y \downarrow x} f(y)$. Analogously we can define $\overline{\lim}_{y \uparrow x} f(y)$ and $\underline{\lim}_{y \uparrow x} f(y)$ and we also have $\underline{\lim}_{y \uparrow x} f(y) \le \overline{\lim}_{y \uparrow x} f(y)$. One can verify as in the sequential case that e.g.

(1) $\overline{\lim}_{y \downarrow x} f(y) \le A$ if and only if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $f(y) < A + \epsilon$ for all y such that $0 < y - x < \delta$.
(2) $\underline{\lim}_{y \downarrow x} f(y) \le A$ if and only if for all $\epsilon > 0$ and $\delta > 0$ there exists an y with $0 < y - x < \delta$ such that $f(y) < A + \epsilon$.

From these and other similar properties one sees that $\lim_{y \to x} f(y) = A$ if and only if $\underline{\lim}_{y \uparrow x} f(y) = \overline{\lim}_{y \uparrow x} f(y) = \underline{\lim}_{y \downarrow x} f(y) = \overline{\lim}_{y \downarrow x} f(y) = A$. Now let $F : (a, b) \to \mathbb{R}$. Then the Dini derivates of F at x are defined as

$$\overline{D^+}F(x) = \overline{\lim_{y \downarrow x}} \frac{F(y) - F(x)}{y - x} = \overline{\lim_{h \downarrow 0}} \frac{F(x + h) - F(x)}{h}$$

$$\underline{D^+}F(x) = \underline{\lim_{y \downarrow x}} \frac{F(y) - F(x)}{y - x} = \underline{\lim_{h \downarrow 0}} \frac{F(x + h) - F(x)}{h}$$

$$\overline{D^-}F(x) = \overline{\lim_{y \uparrow x}} \frac{F(y) - F(x)}{y - x} = \overline{\lim_{h \downarrow 0}} \frac{F(x) - F(x - h)}{h}$$

$$\underline{D^-}F(x) = \underline{\lim_{y \uparrow x}} \frac{F(y) - F(x)}{y - x} = \underline{\lim_{h \downarrow 0}} \frac{F(x) - F(x - h)}{h}$$

From the above we see that $\underline{D^+}F(x) \le \overline{D^+}F(x)$ and $\underline{D^-}F(x) \le \overline{D^-}F(x)$. We say that $F'(x)$ exists if $\underline{D^+}F(x) = \overline{D^+}F(x) = \underline{D^-}F(x) = \overline{D^-}F(x)$ and F is said to be differentiable at x if $F'(x)$ exists and is finite.

Example 1. Define F on $\mathbb{R}$ as follows:

$$F(x) = \begin{cases} |x| & \text{if } x \in \mathbb{Q} \\ |2x| & \text{if } x \notin \mathbb{Q}. \end{cases}$$

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On can check that $\overline{D^+}F(0) = 2$, $\underline{D^+}F(0) = 1$, $\overline{D^-}F(0) = -1$, and $\underline{D^-}F(0) = -2$, while e.g. $\overline{D^+}F(1) = \infty$, $\underline{D^+}F(1) = 1$, $\overline{D^-}F(1) = 1$, and $\underline{D^-}F(1) = -\infty$.

Note that if $\overline{D^+}F(x) > R$, then for all $\delta > 0$ there exists $0 < h < \delta$ such that $\frac{F(x+h)-F(x)}{h} > R$. Similarly $\underline{D^-}F(x) < r$ implies that for all $\delta > 0$ there exists $0 < h < \delta$ such that $\frac{F(x)-F(x-h)}{h} < r$.

2. Vitali covering

Let $E \subset \mathbb{R}$ and $\mathcal{J}$ a collection of intervals. Then $\mathcal{J}$ is called a Vitali covering of E if for all $\epsilon > 0$ and $x \in E$, there exists an interval $I \in \mathcal{J}$ such that $x \in I$ and $0 < |I| < \epsilon$.

Theorem 2. *Let $E \subset \mathbb{R}$ with $m^*(E) < \infty$ and $\mathcal{J}$ a Vitali cover of E. Then for every $\epsilon > 0$ there exist a finite disjoint collection $\{I, \cdots, I_N\}$ of intervals in $\mathcal{J}$ such that*

$$m^*(E \setminus \cup_{n=1}^N I_n) < \epsilon.$$

Proof. We can assume that each interval $I \in \mathcal{J}$ is closed, otherwise we can replace it by its closure $\overline{I}$ and note that $|I| = |\overline{I}|$. Let $O \supset E$ be an open set of finite measure. Then we can assume that $I \subset O$ for all $I \in \mathcal{J}$. Choose $\{I_n\}$ inductively as follows. Choose $I_1 \in \mathcal{J}$ to be any interval, and suppose $I_1, \cdots, I_n$ have already been chosen. Let

$$k_n = \sup\{|I| : I \in \mathcal{J}, I \cap I_k = \emptyset \text{ for } k = 1, \cdots, n\}.$$

Then $I \subset O$ implies $k_n < \infty$. Either $E \subset \cup_{k=1}^n I_k$, or $k_n > 0$ and there exists $I_{n+1} \in \mathcal{J}$ with $|I_{n+1}| > \frac{1}{2}k_n$ and $I_{n+1} \cap I_k = \emptyset$ for $k = 1, \cdots, n$. If this process does not stop, we get a disjoint sequence $\{I_n\}$ in $\mathcal{J}$ with $\sum_{n=1}^\infty |I_n| \le m(O) < \infty$. Hence there exists N such that

$$\sum_{n=N+1}^\infty |I_n| < \frac{\epsilon}{5}.$$

Put $R = E \setminus \cup_{n=1}^N I_n$. To show $m^*(R) < \epsilon$. Let $x \in R$. Then $x \notin \cup_{k=1}^N I_k$, so there exists $I \in \mathcal{J}$ with $x \in I$ and $I \cap I_n = \emptyset$ for $j = 1, \cdots, N$. If $I \cap I_j = \emptyset$ for $j \le n$, then we have $|I| \le k_n < 2|I_{n+1}|$. As $|I_n| \to 0$, there is a smallest n such that $n > N$ and $I \cap I_n \ne \emptyset$. In particular $|I| \le k_{n-1} < 2|I_n|$. Now $x \in I$ and $I \cap I_n \ne \emptyset$ implies that the distance of x to midpoint of I_n is at most $|I| + \frac{1}{2}|I_n| \le \frac{5}{2}|I_n|$. Hence x is in the interval J_n with the same midpoint as I_n and $|J_n| = 5|I_n|$.This shows $R \subset \cup_{N+1}^\infty J_n$, from which we conclude that

$$m^*(R) \le \sum_{N+1}^\infty |J_n| = 5\sum_{N+1}^\infty |I_n| < \epsilon.$$

$\square$

3. The derivative of a monotone function

We start with the crucial lemma.

Lemma 3. *Let $F : [a, b] \to \mathbb{R}$ be an increasing function and let $r < R$. Then the set $E = \{x \in (a, b) : \underline{D^-}F(x) < r < R < \overline{D^+}F(x)\}$ has measure zero.*

Proof. Assume $m^*(E) = s$. Let $\epsilon > 0$. Then there exists an open set $O \supset E$ such that $m(O) < s + \epsilon$. Let $x \in E$. Then $\underline{D^-}F(x) < r$ implies that for all $\delta > 0$ there exists $0 < h < \delta$ such that

$$\frac{F(x) - F(x-h)}{h} < r,$$

i.e., we can find arbitrary small $h > 0$ such that $[x - h, x] \subset O$ and

$$\frac{F(x) - F(x-h)}{h} < r.$$

The collection of all such intervals is a Vitali cover of E, so we can find disjoint intervals $I_1 = [x_1 - h_1, x_1]$, ..., $I_N = [x_N - h_N, x_N]$ such that $m^*(E \setminus \cup_k^N \mathring{I}_k) < \epsilon$. Put $A = E \cap \cup_k^N \mathring{I}_k$. Then $m^*(A) > s - \epsilon$. Moreover, we have

$$\sum_{k=1}^{N} F(x_k) - F(x_k - h_k) < r \sum_{k=1}^{N} h_k < rm(O) < r(s + \epsilon).$$

Let $y \in A$. Then $\overline{D^+}F(y) > R$, so there exist arbitrary small $k > 0$ such that $[y, y+k] \subset I_k$ for some k and for which $F(y+k) - F(y) > Rk$. The collection of such intervals is now a Vitali cover of A, so there exist disjoint $J_1 = [y_1, y_1 + k_1]$, ..., $J_M = [y_M, y_M + k_M]$ of such intervals and $m^*(A \setminus \cup_1^M J_j) < \epsilon$. Then $m^*(A \setminus \cup_1^M J_j) < \epsilon$ implies that $m^*(A \cap \cup_1^M J_j) > s - 2\epsilon$. Summing over the intervals J_j we get that

$$\sum_{j=1}^{M} F(y_j + k_j) - F(y_j) > R \sum k_j > R(s - 2\epsilon).$$

Now each J_j is contained in some I_n, and summing only over those j's for which $J_j \subset I_n$ we get, using the fact that F is increasing, that

$$\sum_{j : J_j \subset I_n} F(y_j + k_j) - F(y_j) \le F(x_n) - F(x_n - h_n).$$

Now summing over all n we get

$$\sum_{n=1}^{N} F(x_n) - F(x_n - h_n) \ge \sum_{j=1}^{M} F(y_j + k_j) - F(y_j) > R(s - 2\epsilon).$$

It follows now that $r(s + \epsilon) > R(s - 2\epsilon)$. This implies $rs \ge Rs$. Now $r < R$ implies that $s = 0$. $\square$

The main theorem is now

Theorem 4. *(Lebesgue) Let $F : [a, b] \to \mathbb{R}$ be an increasing function. Then F is differentiable a.e., F' is measurable, non-negative, and*

$$\int_a^b F' \, dx \le F(b) - F(a).$$

Proof. The set $\{x \in (a,b) : \underline{D^-}F(x) < \overline{D^+}F(x)\} = \cup_{r,R\in\mathbb{Q}}\{x \in (a,b) : \underline{D^-}F(x) < r < R < \overline{D^+}F(x)\}$, which by the above Lemma is a countable union of sets of measure zero. Hence $\underline{D^-}F(x) \geq \overline{D^+}F(x)$ a.e. Now applying this statement to the increasing function $-F(-x)$ instead of $F(x)$, we obtain that $\underline{D^+}F(x) \geq \overline{D^-}F(x)$ a.e. Therefore

$$\overline{D^+}F(x) \leq \underline{D^-}F(x) \leq \overline{D^-}F(x) \leq \underline{D^+}F(x) \leq \overline{D^+}F(x) \text{ for a.e. } x.$$

Hence we conclude that

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

exists a.e. and that F is differentiable where F' is finite. Define $F(x) = F(b)$ for $x > b$ and let

$$G_n(x) = \frac{F(x + 1/n) - F(x)}{1/n}.$$

Then $G_n(x) \geq 0$ and $G_n(x) \to F'(x)$ a.e., which shows that $F' \geq 0$ and measurable. By Fatou's Lemma we have

$$\begin{aligned}
\int_a^b F'\,dx \leq \varliminf_{n\to\infty} \int_a^b G_n(x)\,dx.& \\
&= \varliminf_{n\to\infty} n \int_a^b F(x+1/n) - F(x)\,dx \\
&= \varliminf_{n\to\infty} n \left(\int_{a+1/n}^{b+1/n} F(x)\,dx - \int_a^b F(x)\,dx \right) \\
&= \varliminf_{n\to\infty} n \left(\int_b^{b+1/n} F(x)\,dx - \int_a^{a+1/n} F(x)\,dx \right) \\
&= F(b) - \varlimsup_{n\to\infty} n \int_a^{a+1/n} F(x)\,dx \\
&\leq F(b) - \varlimsup_{n\to\infty} n \int_a^{a+1/n} F(a)\,dx = F(b) - F(a).
\end{aligned}$$

Hence F' is integrable over $[a,b]$ and thus finite a.e., which shows that F is differentiable a.e. on $[a,b]$. $\square$

4. Functions of bounded variation

Let $F : [a,b] \to \mathbb{R}$. Then F is of *bounded variation* over $[a,b]$ if the total variation of F

$$\|F\|_{TV[a,b]} := \sup\{\sum_{k=1}^{n} |F(x_i) - F(x_{i-1})| : a = x_0 < x_1 < \cdots < x_n = b\} < \infty.$$

Note that if F is a monotone function on $[a,b]$, then F is of bounded variation and $\|F\|_{TV[a,b]} = |F(b) - F(a)|$. Similar to the total variation $\|F\|_{TV[a,b]}$ we can define the positive and negative variation of F:

$$\|F\|_{PV[a,b]} := \sup\{\sum_{k=1}^{n} (F(x_i) - F(x_{i-1}))^+ : a = x_0 < x_1 < \cdots < x_n = b\},$$

and

$$\|F\|_{NV[a,b]} := \sup\{\sum_{k=1}^{n}(F(x_i) - F(x_{i-1}))^- : a = x_0 < x_1 < \cdots < x_n = b\}.$$

It is immediate from the definitions that $|F\|_{PV[a,b]} \leq |F\|_{TV[a,b]}$, $|F\|_{NV[a,b]} \leq |F\|_{TV[a,b]}$, and $|F\|_{TV[a,b]} \leq |F\|_{PV[a,b]} + |F\|_{NV[a,b]}$. It will follow from the next proposition that we have in fact an equality in the last inequality.

Proposition 5. *Let $F : [a,b] \to \mathbb{R}$ be of bounded variation. Then the following holds.*

(1) *The functions $x \mapsto ||F\|_{PV[a,x]}$ and $x \mapsto ||F\|_{NV[a,x]}$ are increasing on $[a,b]$.*
(2) *For all $x \in [a,b]$ we have $|F\|_{TV[a,x]} = |F\|_{PV[a,x]} + |F\|_{NV[a,x]}$.*
(3) *For all $x \in [a,b]$ we have $F(x) - F(a) = |F\|_{PV[a,x]} - |F\|_{NV[a,x]}$.*

In particular F can be written as the difference of two increasing functions.

Proof. Let $a = x_0 < x_1 < \cdots < x_n = x$ be a partition of $[a,x]$. Let $x < y \leq b$. Then consider the partition $a = x_0 < x_1 < \cdots < x_n < x_{n+1} = y$ of $[a,y]$. It is clear that

$$\sum_{k=1}^{n}(F(x_i) - F(x_{i-1}))^+ \leq \sum_{k=1}^{n+1}(F(x_i) - F(x_{i-1}))^+,$$

so that $||F\|_{PV[a,x]} \leq ||F\|_{PV[a,y]}$. Similarly we have $||F\|_{NV[a,x]} \leq ||F\|_{NV[a,y]}$ for $x < y$. Now using the identity $a^+ = a^- + a$ we have

$$\sum_{k=1}^{n}(F(x_i) - F(x_{i-1}))^+ = \sum_{k=1}^{n}(F(x_i) - F(x_{i-1}))^- + F(x) - F(a),$$

which implies the inequality $\|F\|_{PV[a,x]} \leq |F\|_{NV[a,x]} + F(x) - F(a)$. Similarly using the inequality $a^- = a^+ - a$ we have

$$\sum_{k=1}^{n}(F(x_i) - F(x_{i-1}))^- = \sum_{k=1}^{n}(F(x_i) - F(x_{i-1}))^+ + F(a) - F(x),$$

so that $\|F\|_{NV[a,x]} \leq |F\|_{PV[a,x]} + F(a) - F(x)$. Combining these two inequalities we get that (3) holds. To prove (2) use the inequality $|a| = 2a^+ - a$ to get

$$\sum_{k=1}^{n}|F(x_i) - F(x_{i-1})| = 2\sum_{k=1}^{n}(F(x_i) - F(x_{i-1}))^+ + F(a) - F(x),$$

which implies $\|F\|_{TV[a,x]} \geq 2\|F\|_{PV[a,b]} + F(a) - F(x) = |F\|_{PV[a,x]} + |F\|_{NV[a,x]}$, by (3). Putting $x = b$ this completes the proof of (2), as we already observed the reverse inequality. □

The following corollary is a consequence of the above proposition.

Corollary 6. *Let $F : [a,b] \to \mathbb{R}$ be of bounded variation. Then F is differentiable a.e., F' is integrable over $[a,b]$ and*

$$\int_a^b |F'(x)|\,dx \leq \|F\|_{TV[a,b]}.$$

Proof. Recall that if G is an increasing function on $[a, b]$ then

$$\int_a^b G'(x)\, dx \le G(b) - G(a).$$

Applying this result to the increasing functions of part (1) of the above proposition we get

$$\int_a^b \left(\|F\|_{PV[a,x]}\right)'\, dx \le \|F\|_{PV[a,b]}$$

and

$$\int_a^b \left(\|F\|_{NV[a,x]}\right)'\, dx \le \|F\|_{NV[a,b]}.$$

Now part (3) of the proposition implies that

$$F'(x) = \left(\|F\|_{PV[a,x]}\right)' - \left(\|F\|_{NV[a,x]}\right)'$$

a.e., so by the above inequalities and (2) of the proposition we have

$$\int_a^b |F'(x)|\, dx \le \|F\|_{PV[a,b]} + \|F\|_{NV[a,b]} = \|F\|_{TV[a,b]}.$$

□

5. Absolute continuous functions

Let $F : [a, b] \to \mathbb{R}$. Then F is called *absolutely continuous* on $[a, b]$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $\{(a_i, b_i) : i = 1, \cdots, n\}$ is a disjoint collection of open intervals in $[a, b]$ with $\sum_{i=1}^n (b_i - a_i) < \delta$, then $\sum_{i=1}^n |F(b_i) - F(a_i)| < \epsilon$.

Note, that by taking $n = 1$ in the above definition, we see immediately that absolute continuity implies uniform continuity.

Lemma 7. *Let f be an integrable function on $[a, b]$. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that if $m(E) < \delta$, then$\int_E |f(t)\, dt < \epsilon$.*

Proof. Let $\epsilon > 0$. Let $f_n = \min(|f|, n)$. Then by the Monotone Convergence Theorem there exists N such that

$$\int |f(t)| - f_N(t)\, dt < \frac{\epsilon}{2}.$$

Then take $\delta = \frac{\epsilon}{2N}$. Then $m(E) < \delta$ implies

$$\int_E |f(t)|\, dt \le \int |f(t)| - f_N(t)\, dt + \int_E f_N(t)\, dt < \frac{\epsilon}{2} + Nm(E) < \epsilon.$$

□

Corollary 8. *Let f be an integrable function on $[a, b]$ and let $F(x) = \int_a^x f(t)\, dt$. Then F is absolutely continuous on $[a, b]$.*

Proof. Let $\epsilon > 0$. Then let $\delta > 0$ as in the above lemma. Let $\{(a_i, b_i) : i = 1, \cdots, n\}$ be a disjoint collection of open intervals in $[a, b]$ with $\sum_{i=1}^{n}(b_i - a_i) < \delta$. Let $E = \cup_{i=1}^{n}(a_i, b_i)$. Then $m(E) < \delta$, so $\int_E |f(t)|\, dt < \epsilon$. This implies that

$$\sum_{i=1}^{n} |F(b_i) - F(a_i)| = \sum_{i=1}^{n} \left| \int_{a_i}^{b_i} f(t)\, dt \right| \leq \sum_{i=1}^{n} \int_{a_i}^{b_i} |f(t)|\, dt = \int_E |f(t)\, dt < \epsilon.$$

□

Proposition 9. *Let $F : [a, b] \to \mathbb{R}$ be absolutely continuous. Then F is of bounded variation.*

Proof. Let $\epsilon = 1$. Then there exists $\delta > 0$ such that if $\{(a_i, b_i) : i = 1, \cdots, n\}$ is a disjoint collection of open intervals in $[a, b]$ with $\sum_{i=1}^{n}(b_i - a_i) < \delta$, then $\sum_{i=1}^{n} |F(b_i) - F(a_i)| < 1$. Let $N \geq 1$ such that $\frac{b-a}{N} < \delta$. Let $a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a, b]$ such that $x_i - x_{i-1} = \frac{b-a}{N}$. Then $\|F\|_{TV[x_{i-1}, x_i]} \leq 1$. Hence $\|T\|_{TV[a,b]} = \sum_{i=1}^{n} \|F\|_{TV[x_{i-1}, x_i]} \leq N < \infty$.

□

The next lemma is the key to the Second Fundamental Theorem of Calculus for the Lebesgue Integral.

Lemma 10. *Let $F : [a, b] \to \mathbb{R}$ be absolutely continuous and assume $F'(x) = 0$ a.e. Then $F(x) = F(a)$ for all $x \in [a.b]$.*

Proof. Let $a < c \leq b$. Then there exists $E \subset (a, c)$ with $m(E) = c - a$ such that $F'(x) = 0$ for all $x \in E$. Let $\epsilon > 0$. Then let $\delta > 0$ be given as by the definition of absolute continuity. Let $x \in E$. Then there exists $h_x > 0$ such that $|F(x+h) - F(x)| < \epsilon h$ for all $0 < h < h_x$. Let $\mathcal{J} = \{[x+h, x] : 0 < h < h_x, x \in E\}$. The $\mathcal{J}$ is a Vitali cover of E. Hence there disjoint intervals $[x_1, x_1 + h_1]$, $\cdots$, $[x_n, x_n + h_n]$ in $\mathcal{J}$ such that

$$m(E \setminus \cup_{i=1}^{n}[x_i, x_i + h_i]) < \delta,$$

so also

$$m((a, c) \setminus \cup_{i=1}^{n}[x_i, x_i + h_i]) < \delta.$$

We can arrange the intervals $[x_i, x_i + h_i]$ so that $x_i + h_i < x_{i+1}$. Then put $a = x_0$ and $b = x_{n+1}$. Then by absolute continuity

$$\sum_{i=0}^{n} |F(x_{i+1}) - F(x_i + h_i)| < \epsilon.$$

By construction of the Vitali cover $\mathcal{J}$ we have

$$\sum_{i=1}^{n} |F(x_i + h_i) - F(x_i)| < \epsilon \sum_{i=1}^{n} h_i \leq \epsilon(c - a).$$

Combining these two estimates we get

$$|F(c) - F(a)| = |\sum_{i=0}^{n}(F(x_{i+1}) - F(x_i + h_i)) + \sum_{i=1}^{n}(F(x_i + h_i) - F(x_i))| \leq \epsilon + \epsilon(c - a)$$

for all $\epsilon > 0$. Hence $F(c) = F(a)$. □

Theorem 11. *(Second Fundamental Theorem of Calculus) Let $F : [a,b] \to \mathbb{R}$. Then F is absolutely continuous $\iff$ F is differentiable a.e. on $[a,b]$, F' is integrable, and $F(x) = F(a) + \int_a^x F'(t)\,dt$.*

Proof. $\Leftarrow$ We have already seen earlier that in this case $G(x) = \int_a^x F'(t)\,dt$ is absolutely continuous and thus F is absolutely continuous.
$\Rightarrow$ If F is absolutely continuous, the F is of bounded variation, so F is differentiable and $F'(x)$ exists a.e. Let $G(x) = \int_a^x F'(t)\,dt$. Then G is absolutely continuous and by the First Fundamental Theorem of Calculus $G'(x) = F'(x)$ a.e. Let $H = F - G$. Then H is absolutely continuous and $H'(x) = 0$ a.e. Hence by the above Lemma we have $H(x) = H(a)$ for all $x \in [a,b]$, i.e.,

$$F(x) - \int_a^x F'(t)\,dt = F(a) - 0 = F(a).$$

□

Corollary 12. *(Lebesgue Decomposition Theorem) Let $F : [a,b] \to \mathbb{R}$ be an increasing function. Then there exist increasing G and H such that $F = G + H$, where G is absolutely continuous and $H'(x) = 0$ a.e. Moreover G and H are unique up to a constant.*

Proof. Homework. □