COMPACT SETS IN METRIC SPACES NOTES FOR MATH 703

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In this note we shall present a proof that in a metric space (X, d) a subset A is compact if and only if it is sequentially compact, i.e., if every sequence in A has a convergent subsequence with limit in A. The usual proofs either use the Lebesgue number of an open cover or reduces an open cover first to a countable cover. For compact subsets of the real line with respect to the Euclidean topology this result has an easier proof that one can easily derive from the equivalence of compactness to being closed and bounded. This theorem is called the Heine-Borel theorem and is usually derived from the theorem that a closed bounded interval is compact. This latter theorem has at least two different proofs. The first one uses the ordering of the real line by considering the supremum of the set of $x \in [a, b]$ such that $[a, x]$ has a finite subcover and showing that this supremum equals b. The other proof is by contradiction. One assumes that one has an open cover without finite subcover and splits the interval into closed subintervals, at least one of which has no finite subcover. Repeating this one gets a sequence of nested closed interval without finite subcovers whose diameters tend to zero, so by the so-called Nested Interval theorem this sequence has an intersection consisting exactly of one point. As this point is covered by an open set of the covering, we can also capture the closed intervals from the nested sequence from some point on, which gives the desired contradiction. Our approach here uses the ideas of this second proof to prove the above mentioned equivalence of compactness to sequential compactness in a metric space. First we recall some standard results.

Lemma 1. *Let (X, d) be a metric space and assume $A \subset X$ is a compact set. Then A is a closed subset of X.*

Proof. We will show that A^c is open. Let $x_0 \in A^c$. Then for all $a \in A$ let $r_a = \frac{1}{2}d(x_0, a)$. Then $B(r_a, x_0) \cap B(r_a, a) = \emptyset$. Now $\{B(r_a, a)\} : a \in A$ is an open covering of A, so there exist $a_1, \cdots, a_n \in A$ such that $A \subset \cup_{k=1}^{n} B(r_{a_k}, a_k)$. Let $r = \min\{r_{a_k} : k = 1, \cdots, n\}$. Then $r > 0$ and $U_r(x_0) \cap B(r_{a_k}, a_k) = \emptyset$ for all $k = 1, \cdots, n$. Hence $U_r(x_0) \cap A = \emptyset$, x_0

is an interior point of A^c. As this holds for all $x_0 \in A^c$ it follows that A^c is open, and thus A is closed. $\square$

Now A is called **totally bounded** if for every $\epsilon > 0$ there exist a finite covering of A consisting of open balls of radius ϵ with centers in A. Clearly a totally bounded set is bounded, but the converse is not true in general.

Proposition 2. *Let (X, d) be a metric space and assume $A \subset X$ is a sequentially compact set. Then A is complete and totally bounded. In particular A is also closed.*

Proof. Let A be sequentially compact. Then every Cauchy sequence in A has a convergent subsequence with limit in A. This implies that such a Cauchy sequence converges to a limit in A, i.e., A is complete and thus also closed. Next let $\epsilon > 0$ and let $x_1 \in A$. If $B(\epsilon, x_1) \supset A$, then we are done. Otherwise let $x_2 \in A \setminus B(\epsilon, x_1)$. If $B(\epsilon, x_1) \cup B(\epsilon, x_2) \supset A$, then we are done. Otherwise let $x_3 \in A \setminus (B(\epsilon, x_1) \cup B(\epsilon, x_2))$. Continuing this way we get either a finite cover of A by balls of radius ϵ, or we a get a sequence (x_n) in A with $d(x_n, x_m) \geq \epsilon$. As such a sequence can't have a convergent subsequence this second possibility does not occur. Thus A is totally bounded. $\square$

Theorem 2.1. *Let (X, d) be a metric space and assume $A \subset X$. Then A is compact if and only if A is a sequentially compact.*

Proof. Assume first that A is compact and let (a_n) be a sequence in A. Let $S = \{a_n : n = 1, 2, \cdots\}$. If the set S is finite, there exists a constant subsequence of (a_n). Assume therefore that S is infinite. Note first that $\overline{S} \subset A$ by the above lemma. We claim there exists $a \in \overline{S}$ such that for every ϵ ball around a there are infinitely many elements of S, which implies directly that there is a subsequence (a_{n_k}) which converges to a. If the claim fails, then for each $a \in \overline{S}$ there exists $r_a > 0$ such that $B(r_a, a) \cap \overline{S} = \{a\}$. Then $\mathcal{C} = \{B(r_a, a) : a \in \overline{S}\} \cup (\overline{S})^c$ is an open cover of A without finite subcover (as S is infinite and each $a \in S$ is covered by exactly one set of $\mathcal{C}$). Thus A is sequentially compact. Assume now that A is sequentially compact and let $\mathcal{C}$ be an open cover of A. Assume that $\mathcal{C}$ has no finite subcover of A. Let $\alpha = \text{diam } (A)$ be the diameter of A. Then by the above proposition A can be covered by finitely many closed balls of radius $\frac{\alpha}{4}$ and with centers in A. At least one of these balls intersected with A can't be finitely covered by $\mathcal{C}$. Call A_1 the intersection of this ball with A. Then A_1 is a closed subset of A with diam $(A_1) \leq \frac{\alpha}{2}$. Repeating now the argument we get a nested sequence of closed sets A_n inside A with diam $(A_n) \leq \frac{\alpha}{2^n}$

such that each A_n can't be finitely covered by $\mathcal{C}$. Let $a_n \in A_n$. Then (a_n) is a Cauchy sequence and by assumption the sequence (a_n) has a convergent subsequence. Hence (a_n) is convergent with limit $a \in A$. As each A_n is closed it follows that $a \in \cap_{k=1}^{\infty} A_n$ and from diam $(A_n) \to 0$ it actually follows that $\{a\} = \cap_{n=1}^{\infty} A_n$. Let $U \in \mathcal{C}$ such that $a \in U$. As U is open there exists $r > 0$ such that $B(r, a) \subset U$. Take now n such that $d(a, a_n) < \frac{r}{2}$ and diam $(A_n) < \frac{r}{2}$. Then $A_n \subset B(r, a) \subset U$, which contradicts that A_n can't be finitely covered by $\mathcal{C}$. □

The above proof was organized in such a way that with only minor modifications it also proves the following theorem.

Theorem 2.2. *Let (X, d) be a metric space and assume $A \subset X$. Then A is compact if and only if A is a complete and totally bounded.*

Proof. Assume first that A is compact. Then by the above theorem A is sequentially compact and thus complete and totally bounded by the proposition preceding that theorem. Alternatively it is easy to prove directly that A is totally bounded and closed. Then one can use the above proof of the implication that A is compact implies A is sequentially compact to produce for given Cauchy sequence in A a convergent subsequence with limit in A, which shows that the Cauchy sequence converges to a limit in A. Assume now that A is complete and totally bounded. Then for a given open cover $\mathcal{C}$ of A without finite subcover, we can construct as above the same sets A_n and Cauchy sequence (a_n), which now converges to a limit in A as A is complete. The rest of the argument is then the same as in the above proof. □

Corollary 1. *Let X be a complete metric space. Then $A \subset X$ is compact if and only if A is closed and totally bounded.*

Proof. This immediate from the above theorem, when we observe that a closed subset of a complete space is complete and that a complete subset of a metric space is closed. □

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