Homework 12 problems.

1. **a.** For $\mu(X) < \infty$, show that

$$L^\infty(X,\mu) \subset L^r(X,\mu) \subset L^p(X,\mu) \subset L^1(X,\mu),$$

where $1 < p < r < \infty$. Show, for $X = (0,1]$, by example that all the inclusions can be strict.

**b.** Show that in general (i.e., if $\mu(X) = \infty$)

$$L^\infty \cap L^1 \subset L^p \subset L^\infty + L^1 = \{f : f = g + h, g \in L^\infty, h \in L^1\}.$$

2. Let $f \in L^2([0,1])$. Prove that

$$\left(\int_{[0,1]} x f(x)\, dx\right)^2 \leq \frac{1}{3} \int_{[0,1]} |f(x)|^2\, dx.$$

3. Let $(X,\mu)$ be a finite measure space and let $1 < p < \infty$. Assume $f_n \in L^p(X,\mu)$ such that $\|f_n\|_p \leq 1$ and $f_n(x) \to 0$ a.e. Prove that $\|f_n\|_1 \to 0$.

4. Let $f_n \to f$ in $L^p$, $1 \leq p < \infty$, and let $\{g_n\}$ be a sequence of measurable functions such that $|g_n| \leq M$ for all $n$, and $g_n \to g$ a.e.

**a.** Prove $\|(g_n - g)f\|_p \to 0$.

**b.** Prove $g_n f_n \to fg$ in $L^p$.