Homework 12 problems.

1. Let $E \subset [0,1] \times [0,1]$ be a measurable set. Assume that $m(E_x) \leq \frac{1}{2}$ for almost every $x \in [0,1]$. Prove that $m(\{y \in [0,1] : m(E^y) = 1\}) \leq \frac{1}{2}$.

2. **a.** For $\mu(X) < \infty$, show that

$$L^\infty(X,\mu) \subset L^r(X,\mu) \subset L^p(X,\mu) \subset L^1(X,\mu),$$

where $1 < p < r < \infty$. Show, for $X = (0,1]$, by example that all the inclusions can be strict.

   b. Show that in general (i.e., if $\mu(X) = \infty$)

$$L^\infty \cap L^1 \subset L^p \subset L^\infty + L^1 = \{f : f = g + h, g \in L^\infty, h \in L^1\}.$$

3. Let $f \in L^2([0,1])$. Prove that

$$\left(\int_{[0,1]} x f(x)\, dx\right)^2 \leq \frac{1}{3} \int_{[0,1]} |f(x)|^2\, dx.$$

4. **a.** Let $f \in L^r \cap L^\infty$ for some $r < \infty$. Prove that

$$\|f\|_p \leq \|f\|_r^{\frac{r}{p}} \|f\|_\infty^{1-\frac{r}{p}}$$

for all $r < p < \infty$.

   b. Asume $f \in L^r \cap L^\infty$ for some $r < \infty$. Prove

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty.$$

(Hint: Use **a.** to get an upper bound for $\limsup_{p\to\infty} \|f\|_p$ and then use that for $0 \leq t < \|f\|_\infty$ the set $A = \{x : |f(x)| \geq t\}$ has positive measure)

5. Let (X,μ) be a finite measure space and let $1 < p < \infty$. Assume $f_n \in L^p(X,\mu)$ such that $\|f_n\|_p \leq 1$ and $f_n(x) \to 0$ a.e. Prove that $\|f_n\|_1 \to 0$.

6. Let $f_n \to f$ in L^p, $1 \leq p < \infty$, and let $\{g_n\}$ be a sequence of measurable functions such that $|g_n| \leq M$ for all n, and $g_n \to g$ a.e.

   a. Prove $\|(g_n - g)f\|_p \to 0$.

   b. Prove $g_n f_n \to fg$ in L^p.