Homework 11 additional problems.

1. Let $F : [a, b] \to \mathbb{R}$ be absolutely continuous. Prove that

$$\int_a^b |F'(x)|\, dx = \|F\|_{TV[a,b]}.$$

2. (Lebesgue decomposition Theorem) Let $F : [a, b] \to \mathbb{R}$ be an increasing function. Prove that there exist increasing G and H such that $F = G + H$, where G is absolutely continuous and $H'(x) = 0$ a.e. Prove also that G and H are unique up to a constant.

3. **a.** Let $F : [0, 1] \to \mathbb{R}$ such that $F'(x)$ exists a.e. and satisfies $F' \in L^1([0, 1])$. Assume F is continuous at 0 and absolutely continuous on $[\epsilon, 1]$ for all $\epsilon > 0$. Prove that F is absolutely continuous on $[0, 1]$.

   b. Let $a > b > 0$ and define $F(0) = 0$, $F(x) = x^a \sin \frac{1}{x^b}$ for $0 < x \leq 1$. Prove that F is absolutely continuous (and thus of bounded variation) on $[0, 1]$.

4. (Quals January 2004)Let f_n be absolutely continuous on $[0, 1]$ and let $f_n(0) = 0$. Assume that

$$\int_0^1 |f_n'(x) - f_m'(x)|\, dx \to 0$$

as $m, n \to \infty$. Prove that f_n converges uniformly to a function f on $[0, 1]$ and that f is absolutely continuous on $[0, 1]$.