Homework 6 additional problems.

1. (a) Prove that if $A \subset E$ is a measurable subset of the non-measurable set E constructed in class (or book), then $m(A) = 0$.

   (b) Assume $A \subset \mathbb{R}$ is with $m^*(A) > 0$. Prove that A contains a non-measurable subset. (Assume first $A \subset [0, 1]$.)

2. (a) Prove that if A, B are closed subsets of $\mathbb{R}$, then $A + B$ is Lebesgue measurable, by showing that $A + B$ is a countable union of compact sets.

   (b) Show that there exist closed A and B with $m(A) = m(B) = 0$, but with $m(A + B) > 0$. (Show that $A = C$, $B = \frac{1}{2}C$ work, where C is the Cantor set)

3. (Cantor like sets of positive measure) Let $0 < \epsilon < 1$. Construct a closed set $\tilde{C}$ as the countable intersection of the closed intervals remaining at the k-th stage of the construction after removing the 2^{k-1} middle open intervals of length $\frac{\epsilon}{2^{2k-1}}$.

   (a) Prove $m(\tilde{C}) = 1 - \epsilon$.

   (b) Prove that for all $x \in \tilde{C}$ there exists $x_n \in [0, 1] \setminus \tilde{C}$ such that $x_n \to x$. Hence $\tilde{C}$ is perfect, and contains no open interval.

   (c) Prove $\tilde{C}$ is uncountable.