# ADDENDUM TO "AND STILL ONE MORE PROOF OF THE RADON–NIKODYM THEOREM."

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In [1] a new proof was given of the Radon–Nikodym Theorem. First for the special case that $0 \leq \nu \leq \mu$ and than the general case was derived via von Neumann's approach, except that the use of Hilbert spaces or the Hahn decomposition theorem was avoided. Based on inquiries I received, it seems that not all readers understand what I mean by the phrase "By an exhaustion argument ...". We will therefore present here a slightly different proof, whereby we will isolate that exhaustion part of the proof as a separate lemma. In fact we shall present two proofs of that argument: one shorter one using the Axiom of Choice, and another constructive one. Moreover we prove now the Radon-Nikodym theorem immediately for the absolutely continuous case and no longer use von Neumann's approach.

**Lemma 1.** *Let $\nu$ and $\mu$ be finite measures on $(X, \mathcal{B})$ such that $\nu \ll \mu$ and $\mu(X) = 1$. Then there exists a measurable set $A$ with $0 < \mu(A)$ such that $\nu(X)\mu(B) \leq \nu(B)$ for all $B \subset A$ and $B \in \mathcal{B}$.*

*Proof.* First (constructive) proof: If $A = X$ fails the conclusion of the lemma, then there exists a smallest integer $n_1 \geq 1$ and a measurable set $A_1$ such that $\nu(X)\mu(A_1) - \nu(A_1) > \frac{1}{n_1}$. By induction, if $X \setminus \cup_{j=1}^{k-1} A_j$ fails the conclusion of the lemma, then there exists a smallest integer $n_k \geq 1$ and a measurable set $A_k \subset X \setminus \cup_{j=1}^{k-1} A_j$ such that $\nu(X)\mu(A_k) - \nu(A_k) > \frac{1}{n_k}$. If the process does not stop for any finite $k$, then we claim that $A = X \setminus \cup_{j=1}^{\infty} A_j$ satisfies the conclusion of the lemma. First observe that $X = A \cup (\cup_j A_j)$ is a disjoint union.Hence

$$0 = \nu(X)\mu(X) - \nu(X) = \nu(X)\mu(A) - \nu(A) + \sum_j (\nu(X)\mu(A_j) - \nu(A_j)).$$

This implies $\mu(A) > 0$ and that that the series converges and thus $\sum_j \frac{1}{n\,j} < \infty$. Hence $n_j \to \infty$. Now let $B \subset A$ be measurable. Then $B \subset X \setminus \cup_{j=1}^{k-1} A_j$ for all $k$ implies that $\nu(X)\mu(B) - \nu(B) \leq \frac{1}{n_k - 1}$ for all $k$ with $n_k > 1$. Hence $\nu(X)\mu(B) - \nu(B) \leq 0$, which concludes the proof of the lemma.

Second proof: Assume that the lemma fails. Then for all $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $B \in \mathcal{B}$ such that $\nu(X)\mu(B) > \nu(B)$ (*). By Zorn's Lemma there exists a disjoint collection $\{B_n\}$ such that each $B_n$ satisfies (*) and $\mu(X \setminus \cup_n B_n) = 0$. This implies that also $\nu(X \setminus \cup_n B_n) = 0$. Hence

$$\nu(X) = \sum_n \nu(B_n) < \sum_n \nu(X)\mu(B_n) = \nu(X).$$

This is a contradiction and the lemma follows. □

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**Theorem 2.** *(Radon-Nikodym) Let $\nu$ and $\mu$ be finite measures on $(X,\mathcal{B})$ with $\nu \ll \mu$. Then there exists a measurable function $f_0$ with $0 \le f_0$ such that $\nu(E) = \int_E f_0 \, d\mu$ for all $E$ in $\mathcal{B}$.*

*Proof.* Without loss of generality we can assume that $\mu(X) = 1$. Let $H = \{f : f \text{ measurable}, 0 \le f, \int_E f \, d\mu \le \nu(E) \text{ for all } E \in \mathcal{B}\}$. Note that $H \neq \emptyset$, since 0 belongs to $H$. Moreover, when $f_1, f_2 \in H$, then $\max\{f_1, f_2\} \in H$. Indeed, if $A = \{x : f_1(x) \ge f_2(x)\}$ and $B = A^c$, then

$$\begin{aligned}\int_E \max\{f_1,f_2\}\,d\mu &= \int_{E\cap A} \max\{f_1,f_2\}\,d\mu + \int_{E\cap B} \max\{f_1,f_2\}\,d\mu \\ &= \int_{E\cap A} f_1\,d\mu + \int_{E\cap B} f_2\,d\mu \le \nu(E\cap A) + \nu(E\cap B) = \nu(E).\end{aligned}$$

Let $M = \sup\{\int f \, d\mu : f \in H\}$. Then $0 \le M \le \nu(X) < \infty$, so there exist functions $f_n$ in $H$ with $f_1 \le f_2 \le \dots$ such that $\int f_n \, d\mu > M - \frac{1}{n}$. Let $f_0 = \lim f_n$. Then $f_0$ is measurable. By the Monotone Convergence Theorem, $f_0 \in H$ and $\int f_0 \, d\mu \ge M$. Hence $\int f_0 \, d\mu = M$. To complete the proof we show that $\nu(E) = \int_E f_0 \, d\mu$. Suppose $\nu(E) > \int_E f_0 \, d\mu$ for some $E$ in $\mathcal{B}$. Then $\nu_1$ defined by $\nu_1(E) = \nu(E) - \int_E f_0 \, d\mu$ is a finite measure with $\nu_1(X) > 0$, which satisfies the hypothesis of the previous lemma. Let $A$ be as in the conclusion of the lemma. Then $f_0 + \nu_1(X)\chi_A \in H$ as can be seen as follows

$$\begin{aligned}\int_E f_0 + \nu_1(X)\chi_A\,d\mu &= \int_E f_0\,d\mu + \nu_1(X)\mu(A\cap E) \\ &\le \int_E f_0\,d\mu + \nu(A\cap E) - \int_{A\cap E} f_0\,d\mu \\ &= \int_{E\cap A^c} f_0\,d\mu + \nu(A\cap E) \le \nu(E\cap A^c) + \nu(E\cap A) = \nu(E).\end{aligned}$$

Moreover $\int f_0 + \nu_1(X)\chi(A) \, d\mu = M + \nu_1(X)\mu(A) > M$, which contradicts the definition of $M$. □

**Remark.** The extension to the $\sigma$-finite case is routine.

## References

[1] Anton R. Schep, And still one more proof of the Radon–Nikodym theorem, *Amer. Math. Monthly* **110**(2003) 526–538.

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