A NEWER ADDENDUM TO "AND STILL ONE MORE PROOF OF THE RADON–NIKODYM THEOREM."

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In [1] a new proof was given of the Radon–Nikodym Theorem. First for the special case that $0 \le \nu \le \mu$ and than the general case was derived via von Neumann's approach, except that the use of Hilbert spaces or the Hahn decomposition theorem was avoided. Based on inquiries I received, it seems that not all readers understand what I mean by the phrase "By an exhaustion argument ...". We will therefore present here a detailed proof of this statement, as a separate lemma. In fact we shall present two proofs of that argument: one shorter one using the Axiom of Choice, and another constructive one. Moreover we prove now the Radon-Nikodym theorem immediately for the absolutely continuous case and no longer use von Neumann's approach.

Lemma 1. *Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and let (P) be some property which any measurable sets does or does not possess. Assume that for all $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $B \subset A$ in $\mathcal{B}$ with property (P) and $\mu(B) > 0$. Then there exists a disjoint collection (B_n) in $\mathcal{B}$ such that $\mu(X \setminus \cup_n B_n) = 0$ and such that each B_n has property (P). In particular, if the property is preserved under countable disjoint unions and under change by sets of measure zero, then X has property (P).*

Proof. It is clear that it suffices to prove the lemma for the case that $\mu(X) < \infty$.
First (non-constructive) proof: Let $\mathcal{A}$ be the set of all disjoint sets (B_i) with $B_i \in \mathcal{B}$, $\mu(B_i) > 0$ and such that each B_i has property (P). We can partial order the set $\mathcal{A}$ by set theoretical inclusion, i.e., $(B_i) \le (C_j)$ if $(B_i) \subset (C_j)$. Then every chain in $\mathcal{A}$ has a (least) upper bound, namely the union of all the elements of the chain. Hence by Zorn's lemma $\mathcal{A}$ has a maximal element $(B_i)_{i \in I}$. Since $\mu(B_i) > 0$ and $\mu(X) < \infty$ the index set I is countable. The maximality ensures now that $\mu(X \setminus \cup_i B_i) = 0$, as otherwise we can find $B \subset X \setminus \cup_i B_i$ with $\mu(B) > 0$ and such that B has property (P).
Second (constructive) proof: To avoid having to divide by $\mu(X)$ we make the further assumption that $\mu(X) = 1$. This can be accomplished by replacing the measure μ by the measure $\frac{1}{\mu(X)}\mu$. Let $\alpha_1 = \sup\{\mu(B) : B \text{ has property } (P)\}$. Let n_1 be the smallest natural number n such that $\frac{1}{n} < \alpha_1$. Then there exists B_1 with property (P) such that $\frac{1}{n_1} < \mu(B_1)$. By induction, if $X \setminus \cup_{j=1}^{k-1} B_j$ has positive measure for all $k \ge 2$ we let $\alpha_k = \sup\{\mu(B) : B \subset X \setminus \cup_{j=1}^{k-1} B_j, B \text{ has property } (P)\}$. Then $\alpha_k > 0$. Let n_k be the smallest natural number n such that $\frac{1}{n} < \alpha_k$. Then there exists $B_k \subset X \setminus \cup_{j=1}^{k-1} B_j$ with property (P) such that $\frac{1}{n_k} < \mu(B_k)$. Note that if $n_{k-1} < n_k$, then $n_{k-1} < \frac{1}{\alpha_k} \le n_k$, which implies that in that case $\mu(B) \le \frac{1}{n_{k-1}}$ for

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all $B \subset X \setminus \cup_{j=1}^{k-1} B_j$. Now we have

$$\sum_{k=1}^{\infty} \frac{1}{n_k} \leq \sum_{k=1}^{\infty} \mu(B_k) \leq \mu(X) = 1.$$

Hence $n_k \to \infty$ as $k \to \infty$. We claim that $\mu(X \setminus \cup_{k=1}^{\infty} B_k) = 0$. If not, then there exists $B \subset X \setminus \cup_{j=1}^{\infty} B_j$ with $\mu(B) > 0$ such that B has property (P). Then $B \subset X \setminus \cup_{j=1}^{k-1} B_j$ for all k and $n_{k-1} < n_k$ for infinitely k together imply that $\mu(B) \leq \frac{a}{n_{k-1}}$ for infinitely many k. As $n_k \to \infty$ as $k \to \infty$ it follows that $\mu(B) = 0$. This is a contradiction and the lemma follows. $\square$

Theorem 2. *(Radon-Nikodym) Let ν and μ be measures on $(X, \mathcal{B})$ with $\nu \ll \mu$. Assume μ is σ-finite and that ν is a finite measure.Then there exists an integrable function $f_0 \geq 0$ such that $\nu(E) = \int_E f_0 \, d\mu$ for all E in $\mathcal{B}$.*

Proof. Let $H = \{f : f \text{ measurable}, 0 \leq f, \int_E f \, d\mu \leq \nu(E) \text{ for all } E \in \mathcal{B}\}$. Note that $H \neq \emptyset$, since 0 belongs to H. Moreover, when $f_1, f_2 \in H$, then $\max\{f_1, f_2\} \in H$. Indeed, if $A = \{x : f_1(x) \geq f_2(x)\}$ and $B = A^c$, then

$$\begin{aligned}\int_E \max\{f_1, f_2\} \, d\mu &= \int_{E \cap A} \max\{f_1, f_2\} \, d\mu + \int_{E \cap B} \max\{f_1, f_2\} \, d\mu \\ &= \int_{E \cap A} f_1 \, d\mu + \int_{E \cap B} f_2 \, d\mu \leq \nu(E \cap A) + \nu(E \cap B) = \nu(E).\end{aligned}$$

Let $M = \sup\{\int f \, d\mu : f \in H\}$. Then $0 \leq M \leq \nu(X) < \infty$, so there exist functions f_n in H with $f_1 \leq f_2 \leq \ldots$ such that $\int f_n \, d\mu > M - \frac{1}{n}$. Let $f_0 = \lim f_n$. Then f_0 is measurable. By the Monotone Convergence Theorem, $f_0 \in H$ and $\int f_0 \, d\mu \geq M$. Hence $\int f_0 \, d\mu = M$. To complete the proof we show that $\nu(E) = \int_E f_0 \, d\mu$. Suppose $\nu(E_0) > \int_{E_0} f_0 \, d\mu$ for some E_0 in $\mathcal{B}$. Then $\mu(E_0) > 0$, since $\nu(E_0) > 0$. Let $\epsilon > 0$ such that $\int_{E_0} f_0 + \epsilon \, d\mu < \nu(E_0)$. Let (P) be the property $\int_E f_0 + \epsilon \chi_{E_0} \, d\mu > \nu(E)$. Then E_0 does not have property (P) and property (P) is preserved under countable disjoint unions. Moreover by absolute continuity of ν w.r.t. μ it follows that (P) is preserved under change by sets of μ-measure zero. We claim there exists a measurable $F \subset E_0$ with $\mu(F) > 0$ such that $f_0 + \epsilon \chi_F \in H$. If not, then for all measurable $F \subset E_0$ of positive μ-measure there exists a measurable $G \subset F$ with $\int_G f_0 + \epsilon \chi_G \, d\mu > \nu(G)$. Hence for each measurable $F \subset E_0$ of positive μ-measure there exists a measurable $G \subset F$ with $\int_G f_0 + \epsilon \chi_{E_0} \, d\mu > \nu(G)$, i.e., each measurable $F \subset E_0$ of positive μ-measure contains a measurable $G \subset F$ with property (P). By the previous Lemma E_0 has property (P), which is a contradiction. Hence there exists a measurable $F \subset E_0$ with $\mu(F) > 0$ such that $f_0 + \epsilon \chi_F \in H$. Now $\int f_0 + \epsilon \chi_F \, d\mu = M + \epsilon \mu(F) > M$, which contradicts the definition of M and the proof is complete. $\square$

Remark. The extension to the σ-finite case is routine.

References

[1] Anton R. Schep, And still one more proof of the Radon–Nikodym theorem, *Amer. Math. Monthly* **110**(2003) 526–538.

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