THE OPEN MAPPING THEOREM AND RELATED THEOREMS

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We start with a lemma, whose proof contains the most ingenious part of Banach's open mapping theorem. Given a norm $\|\cdot\|_i$ we denote by $B_i(x,r)$ the open ball $\{y \in X : \|y-x\|_i < r\}$.

Lemma 1. *Let X be a vector space with two norms $\|\cdot\|_1$, $\|\cdot\|_2$ such that $(X, \|\cdot\|_1)$ is a Banach space and assume that the identity map $I : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is continuous. If $B_2(0,1) \subset \overline{B_1(0,r)}^{\|\cdot\|_2}$, then $B_2(0,1) \subset B_1(0,2r)$ and the two norms are equivalent.*

Proof. From the hypothesis we get $B_2(0,1) \subset B_1(0,r) + B_2(0,\frac{1}{2})$, so by scaling we get that $B_2(0,\frac{1}{2^n}) \subset B_1(0,\frac{r}{2^n}) + B_2(0,\frac{1}{2^{n+1}})$ for all $n \geq 1$. let now $\|y\|_2 < 1$. Then we can write $y = x_1 + y_1$, where $\|x_1\|_1 < r$ and $\|y_1\|_2 < \frac{1}{2}$. Assume we have $\|y_n\|_2 < \frac{1}{2^n}$ we can write $y_n = x_{n+1} + y_{n+1}$, where $\|x_{n+1}\|_1 < \frac{r}{2^n}$ and $\|y_{n+1}\|_2 < \frac{1}{2^{n+1}}$. By completeness of $(X, \|\cdot\|_1)$ there exists $x \in X$ such that $x = \sum_{n=1}^{\infty} x_n$, where the series converges with respect to the norm $\|\cdot\|_1$. By continuity of the identity map $I : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ it follows that the same series also converges to x with respect to $\|\cdot\|_2$. On the other hand the equation $y = \sum_{k=1}^{n+1} x_k + y_{n+1}$ shows that the series $\sum_{n=1}^{\infty} x_n$ converges to y with respect to $\|\cdot\|_2$. Hence $y = x$ and thus $\|y\|_1 = \|x\|_1 \leq \sum_{n=1}^{\infty} \|x_n\|_1 < 2r$. It follows that$B_2(0,1) \subset B_1(0,2r)$ and thus $\|y\|_1 \leq 2r\|y\|_2$ for all $y \in X$. As the continuity of I gives that there exists C such that $\|y\|_2 \leq C\|y\|_1$ for all $y \in X$, we get that the two norms are equivalent.

□

Theorem 2. *Let X be a vector space with two norms $\|\cdot\|_1$, $\|\cdot\|_2$ such that $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are Banach spaces. Assume that the identity map $I : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2)$ is continuous. Then the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.*

Proof. Applying the Baire Category theorem in $(X, \|\cdot\|_2)$ to $X = \cup_{n=1}^{\infty} B_1(0,n)$ we can find n_0, x_0 and $r_0 > 0$ such that $B_2(x_0,r_0) \subset \overline{B_1(0,n_0)}^{\|\cdot\|_2}$. Translating over $-x_0$ we get that $B_2(0,r_0) \subset \overline{B_1(-x_0,n_0)}^{\|\cdot\|_2}$. Now by the triangle inequality we get that $B_2(0,r_0) \subset \overline{B_1(0,n_0+\|x_0\|_1)}^{\|\cdot\|_2}$. By the above lemma the two norms are equivalent. □

Theorem 3 (Bounded Inverse Theorem)**.** *Let X, Y be Banach spaces and assume $T : X \to Y$ is an one-to-one, onto continuous linear operator. Then $T^{-1} : Y \to X$ is continuous.*

Proof. Define $\|y\|_T = \|T^{-1}y\|$. Then $\|\cdot\|_T$ is a norm on Y and $\|y\| \leq \|T\|\|y\|_T$ for $y \in Y$, so $I : (Y, \|\cdot\|_T) \to (Y, \|\cdot\|)$ is continuous. Moreover, if $\sum_{n=1}^{\infty} \|y_n\|_T < \infty$, then $x_0 = \sum_{n=1}^{\infty} T^{-1}y_n$ exists in X and $\|Tx_0 - \sum_{n=1}^{N} y_n\|_T = \|x_0 - \sum_{n=1}^{N} T^{-1}y_n\| \to 0$ as $N \to \infty$. Hence $(Y, \|\cdot\|_T)$ is also a Banach space. By the above Theorem the two norms on Y are equivalent, so there exists C such that $\|T^{-1}(y)\| \leq C\|y\|$ for all $y \in Y$, i.e. T^{-1} is continuous. $\square$

We recall now that a linear map $T : X \to Y$ is called *open* if $T(O)$ is open for all open $O \subset X$. It is easy to see that an open linear map is surjective. The Open Mapping theorem gives a converse to that statement. Before stating and proving that theorem, we recall a few basic facts about quotient maps. Let X be a Banach space and $M \subset X$ a closed subspace. Then X/M is a Banach space with respect to the quotient norm $\|[x]\| = \inf\{\|y\|; y \in [x]\}$. Denote by Q the quotient map $Q(x) = [x]$. Then Q is open. In fact, it is easy to see from the definition of the quotient norm that $Q(\{x : \|x\| < 1\}) = \{[x] : \|[x]\| < 1\}$.

Theorem 4 (Open Mapping Theorem). *Let X, Y be Banach spaces and assume $T : X \to Y$ is an onto continuous linear operator. Then T is an open map.*

Proof. Let $Q : X \to X/\ker(T)$ be the quotient map. Then by the above remarks Q is an open mapping. Let $\hat{T} : X/\ker(T) \to Y$ be the induced map such that $T = \hat{T} \circ Q$. Then $\hat{T}$ is one to one and onto, so by the above Theorem $\hat{T}^{-1}$ is continuous, so $\hat{T}$ is open and thus T is open.

$\square$

Let now $A : \mathcal{D}(A) \to Y$ be a linear operator, where $\mathcal{D}(A)$ is a (not necessarily closed) linear subspace of the Banach space X. The subspace $\mathcal{D}(A)$ is called the domain of A. Given a linear operator $A : \mathcal{D}(A) \to Y$ we define the graph

$$\Gamma(A) = (x, Ax) : x \in \mathcal{D}(A),$$

It is clear that $\Gamma(A)$ is linear subspace of $X \times Y$. We can equip $X \times Y$ with the product norm $\|(x, y)\| = \|x\| + \|y\|$. Then we say that A has a closed graph (or is a closed operator), if $\Gamma(A)$ is a closed subspace of $X \times Y$.

Example 5. *Let $X = Y = C[0, 1]$ with the supremum norm. Let $\mathcal{D}(A) = C'[0, 1]$ the subspace of X consisting of continuously differentiable functions and define $A : \mathcal{D}(A) \to Y$ by $Af = f'$. One can can see that A is not bounded, by taking $f_n(t) = t^n$, and noting that $\|f_n\| = 1$ and $\|Af_n\| = n$. On the other hand A has a closed graph. To see that A has a closed graph, let $(f_n, f_n') \to (f, g)$ in $X \times Y$. Then by the Fundamental Theorem of Calculus $f_n(t) - f_n(0) = \int_0^t f_n'(s)\,ds \to \int_0^s g(s)\,ds$. It follows that $f(t) = f(0) + \int_0^t g(s)\,ds$. Hence $f \in \mathcal{D}(A)$ and $f' = g$, i.e., $(f, g) \in \Gamma(A)$.*

The following proposition is immediate from the definition.

Proposition 6. *Let X and Y be Banach spaces and assume $A : \mathcal{D}(A) \to Y$ is a linear operator, where $\mathcal{D}(A)$ is a subspace of X. Then the following are equivalent.*

1. *A has a closed graph.*
2. *If $x_n \in \mathcal{D}(A)$, $x_n \to x \in X$, and $Ax_n \to y \in Y$, then $x \in \mathcal{D}(A)$ and $Ax = y$.*
3. *$\mathcal{D}(A)$ is a Banach space with respect to the graph norm $\|x\|_A = \|x\| + \|Ax\|$.*

Theorem 7 (Closed Graph Theorem). *et X and Y be Banach spaces and assume $A : X \to Y$ is a closed linear operator. Then A is bounded.*

Proof. Define $P : \Gamma(A) \to X$ by $P(x, Ax) = x$. Then P is clearly a bounded, one-to-one, onto linear operator, so by the Bounded Inverse Theorem the inverse operator $P^{-1} : X \to \Gamma(A)$ is bounded. Hence there exists a constant C such that $\|x\| + \|Ax\| \leq C\|x\|$, i.e., $\|Ax\| \leq (C-1)\|x\|$ for all $x \in X$.

□

We now present a proof of the Uniform Boundedness Principle, based on the Closed Graph Theorem.

Theorem 8 (Banach-Steinhaus)**.** *Let X and Y be Banach spaces and assume $A_\alpha \in L(X, Y)$ $(\alpha \in \mathcal{F})$ is a pointwise bounded family of bounded operators, i.e., for all $x \in X$ there exists a constant C_x such that $\|A_\alpha x\| \leq C_x$ for all $\alpha \in \mathcal{F}$. Then there exists a constant C such that $\|A_\alpha\| \leq C$ for all $\alpha \in \mathcal{F}$.*

Proof. Define the space $\oplus_\alpha Y = \{(y_\alpha) : y_\alpha \in Y, \sup_\alpha \|y_\alpha\| < \infty\}$ with norm $\|(y_\alpha)\| = \sup_\alpha \|y_\alpha\|$. It is straightforward to verify that $\oplus_\alpha Y$ is also a Banach space. Now define $T : X \to \oplus_\alpha Y$ by $Tx = (A_\alpha x)$. Note $Tx \in \oplus_\alpha Y$, since the collection A_α is pointwise bounded. Clearly T is linear and we claim that T is closed. To see this, let $x_n \to 0$ and $Tx_n \to (y_\alpha)$. Then $A_\alpha x_n \to y_\alpha$ for all $\alpha \in \mathcal{F}$, but also $A_\alpha x_n \to 0$ for all α. Hence $(y_\alpha) = (0)$ and T is closed. By the Closed Graph Theorem T is bounded, i.e., there exists a constant C such that for all $\|x\| \leq 1$ we have $\sup_\alpha \|A_\alpha x\| \leq C$. Hence $\|A_\alpha\| \leq C$ for all $\alpha \in \mathcal{F}$. □