Make-up Problems for Test 2, MATH 554/703I

Instructions: You can not collaborate or use a tutor to do these problems. You do not have to do all problems to get extra credit. Provide detailed answers justifying all steps to get full credit (especially for problem 3). There are in total 20 points to be gained.

(1) Let $A \subset \mathbb{R}$ a given non-empty set which is bounded above and let $a = \text{lub } A$. Assume $a \notin A$. Prove that a is a limit point of A.

(2) Prove that every uncountable subset of $\mathbb{R}$ has a limit point in $\mathbb{R}$. (Hint: write $A = \cup A_n$, where $A_n = A \cap [-n, n]$.)

(3) Find the lim inf and lim sup of the following sequence

$$\left\{ \frac{n + (-1)^n n^2}{n^2 + 1} \right\}.$$

(4) Let $\{a_n\}$ be a convergent sequence in $\mathbb{R}$ with $a = \lim a_n$. Prove, using the definition (i.e. you can't use the Heine-Borel theorem), that $A = \{a\} \cup \{a_n : n = 1, 2, \cdots\}$ is compact.

(5) Let $\{A_k\}$ be a sequence of compact sets. Assume that

$$A_1 \cap \cdots \cap A_n \neq \emptyset$$

for all $n \in \mathbb{N}$. Prove that $\cap_{k=1}^{\infty} A_k \neq \emptyset$. Note that you can't assume that the A_k's contain closed intervals, but that you will either have to use the definition or the "sequential" compactness of the sets.