Final HW problems.

Hand these in only by the end of the semester or Final exam time. No collaboration on these problems. You can ask me for hints. I don't expect you to do them all, but expect a serious effort on some.

1. Consider the quotient space ℓ_∞/c_0. Prove that
$$\|[x]\| = \limsup |x_n|.$$

2. Let $\{e_n : n = 1, 2, \cdots\}$ denote the standard orthonormal basis of ℓ_2. Let $E_1 = \overline{\mathrm{span}}\{e_{2n-1}; n = 1, 2, \cdots\}$ and $E_2 = \overline{\mathrm{span}}\{e_{2n-1} + \frac{1}{n}e_{2n}; n = 1, 2, \cdots\}$. Prove
   (a) $E_1 \cap E_2 = \{0\}$.
   (b) $E_1 \oplus E_2$ is dense, but not closed in ℓ_2.

3. Let H be an inner product space. Assume $H = M \oplus M^\perp$ for all closed subspaces M. Prove H is complete.

4. Let X, Y, Z Banach spaces and $A : X \to Y$ and $B : Y \to Z$ linear maps with B bounded and injective and BA bounded. Prove that A is bounded too.

5. Let X be an infinite dimensional Banach space. Prove that every Hamel basis (i.e. maximal linearly independent system) in X is uncountable. (Hint: Assume the contrary, write X as a countable union of closed subspaces and apply the Baire Category Theorem).

6. Let X be a Banach space and let $T_n \in L(X)$ such that $\lim_{n\to\infty}]\|T_n x\|^{\frac{1}{n}} = 0$. Prove that $\lim_{n\to\infty} \|T_n\|^{\frac{1}{n}} = 0$. (Hint: Use the Baire Category Theorem.)

7. Let X, Y be Banach spaces. Let $T : X \to Y$ be a compact linear operator and let $x_n \in X$ converge weakly to 0. Prove that $\|Tx_n\| \to 0$.