SUPPLEMENTAL NOTES FOR MATH 704.
SPRING 2012

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1. Classical Banach Spaces

1.1. **Normed spaces.** Recall that a (real) vector space V is called a **normed space** if there exists a function $\|\cdot\| : V \to \mathbb{R}$ such that

(1) $\|f\| \geq 0$ for all $f \in V$ and $\|f\| = 0$ if and only if $f = 0$.
(2) $\|af\| = |a| \, \|f\|$ for all $f \in V$ and all scalars a.
(3) (Triangle inequality) $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in V$.

If V is a normed space, then $d(f, g) = \|f-g\|$ defines a metric on V. Convergence w.r.t this metric is called *norm convergence*. If V is a complete metric space w.r.t. this metric, then V is called a **Banach space**.

Let $(X, \mathcal{B}, \mu)$ be a measure space and $0 < p \leq \infty$. Then we shall first define $\mathcal{L}^p(X, \mu)$, where we will usually write $\mathcal{L}^p$ and omit X and μ. For $0 < p < \infty$ we define

$$\mathcal{L}^p(X, \mu) = \{f : f : X \to \mathbb{R} \cup \{\pm\infty\} \text{ measurable }, \int_X |f|^p \, d\mu < \infty\},$$

and for $p = \infty$ we define

$$\mathcal{L}^\infty(X, \mu) = \{f : f : X \to \mathbb{R} \cup \{\pm\infty\} \text{ measurable}, |f(x)| \leq M \text{ a.e. for some } M\}.$$

Functions in $\mathcal{L}^p$ with $0 < p < \infty$ are called pth-integrable and functions in $\mathcal{L}^\infty$ are called essentially bounded functions. Note that if $f, g \in \mathcal{L}^p$ for $0 < p \leq \infty$ implies that f and g are finite a.e.. so that $f + g$ is well-defined except for set of measure zero. Therefore we introduce the relation $\sim$ on $\mathcal{L}^p$ with $0 < p \leq \infty$ by defining $f \sim g$ if $f(x) = g(x)$ a.e.. It is immediate that $\sim$ is an equivalence relation on each $\mathcal{L}^p$ with $0 < p \leq \infty$. Now we define $L^p(X, \mu)$, or just L^p, as the set of equivalence classes $\mathcal{L}^p/\sim$, i.e., elements of L^p are functions, where we identify functions equal a.e.. Therefore we will continue to refer to elements of L^p as functions and denote as before by f and g. Let now $0 < p < \infty$ and $f, g \in \mathcal{L}^p$. Then as remarked above $f + g$ is well-defined a.e. and is again a measurable function and $|f + g|^p \leq (2\max(|f|, |g|))^p \leq 2^p \max(|f|^p, |g|^p) \leq 2^p(|f|^p + |g|^p)$ implies that $f + g \in \mathcal{L}^p$. This implies that if $f, g \in L^p$, then $f + g \in L^p$. In case $p = \infty$ and $f, g \in \mathcal{L}^\infty$, then $|f(x)| \leq M_1$ a.e. and $|g(x)| \leq M_2$ a.e implies that $|f(x) + g(x)| \leq M_1 + M_2$ a.e., so that $f + g \in \mathcal{L}^\infty$. Hence $f, g \in L^\infty$ implies that $f + g \in L^\infty$. It is also straightforward to show that if $f \in L^p$, $0 < p \leq \infty$, then $af \in L^p$ for all scalars a. As addition and scalar multiplication are defined a.e. pointwise on L^p it immediate that they satisfy the axioms of a vector space addition and scalar multiplication. Hence we have proved

Proposition 1. *For $0 < p \leq \infty$ the set L^p is a vector space.*

Date: April 9, 2012.

2. The Banach space L^∞

For $f \in L^\infty$ define

$$\|f\|_\infty = \inf\{M : |f(x)| \le M \text{ a.e.}\}.$$

Proposition 2. *For $f \in L^\infty$ we have $|f(x)| \le \|f\|_\infty$ a.e.*

Proof. Let $M_1 \ge M_2 \ge \cdots \downarrow \|f\|_\infty$ such that $|f(x)| \le M_n$ a.e.. Then the set $E_n = \{x : |f(x)| > M_n\}$ has measure zero for all n. Hence $\mu(\cup_n E_n) = 0$. If $x \notin \cup_n E_n$, then $|f(x)| \le M_n$ for all $n \ge 1$ and thus $|f(x)| \le \|f\|_\infty$. It follows that $|f(x)| \le \|f\|_\infty$ a.e. □

Note that it follows from the definition of $\|f\|_\infty$ and the above theorem that also

$$\|f\| = \inf\{\sup |g(x)| : f = g \text{ a.e.}\}.$$

For this reason $\|f\|_\infty$ is called the essential supremum of f.

Theorem 3. *The vector space L^∞ is a Banach space w.r.t. to $\|f\|_\infty$.*

Proof. First we show that $\|f\|_\infty$ is a norm on L^∞. Clearly $\|f\|_\infty \ge 0$ and from the above proposition it follows that $\|f\|_\infty = 0$ if and only if $f = 0$ a.e.. From $|f(x)| \le M$ if and only $|af(x)| \le |a|M$ it follows that $\|af\|_\infty = |a|\|f\|_\infty$. For the triangle inequality we observe that $|f(x) + g(x)| \le |f(x)| + |g(x)| \le \|f\|_\infty + \|g\|_\infty$ a.e., so that $\|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty$. Hence L^∞ is a normed space. It remains to show that L^∞ is complete. Let $\{f_n\}$ be a norm Cauchy sequence in L^∞. Then we can find a measurable set E with $\mu(E^c) = 0$ such that for all $x \in E$ we have $|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty$ for all m, n. This implies that $\{f_n(x)\}$ is a (uniform) Cauchy sequence on E. Define $f(x) = \lim_n f_n(x)$ for $x \in E$ and define $f(x) = 0$ on E^c. Then f measurable and $f_n - f$ converges uniformly to zero on E. Hence $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. Remains to show that $f \in L^\infty$, but this follows immediately from $|f(x)| \le |f_n(x)| + |f(x) - f_n(x)| \le \|f_n\|_\infty + \|f - f_n\|_\infty$ a.e. □

3. The Banach space L^p, $1 \le p < \infty$

For $1 \le p < \infty$ we define

$$\|f\|_p = \left(\int |f|^p \, dx \right)^{\frac{1}{p}}.$$

It is immediate that $\|f\|_p \ge 0$ and $\|f\|_p = 0$ if and only if $f = 0$ a.e. Also clear is that $\|af\|_p = |a|\|f\|_p$ for all scalars a and $f \in L^p$. For $p = 1$ it is obvious from $\int |f + g| \, dx \le \int |f| \, dx + \int |g| \, dx$ that $\|f\|_1$ is a norm on L^1. For $p > 1$ the triangle inequality is less obvious and we will prove first another very useful inequality. First we recall

Lemma 4. *(Young's inequality) Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $a, b \ge 0$ we have*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

Moreover, equality holds if and only if $a^p = b^q$.

Theorem 5. *(Hölders Inequality) Let $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and*

$$\int |fg|\, dx \le \|f\|_p \|g\|_q.$$

For $1 < p < \infty$ equality holds if and only if there exists $(\alpha, \beta) \ne (0,0)$ such that $\alpha |f|^p = \beta |g|^q$.

Proof. If $p = 1$ and $q = \infty$, then $|fg| \le \|g\|_\infty |f|$ a.e. implies $fg \in L^1$ and $\int |fg|\, dx \le \|g\|_\infty \int |f|\, dx = \|f\|_1 \|g\|_\infty$. The proof is similar for $p = \infty$ and $q = 1$. Assume therefore $1 < p < \infty$. Assume first that $\|f\|_p = 1$ and $\|g\|_q = 1$. Then take $a = |f(x)|$ and $b = |g(x)|$ in Young's inequality to get

$$|f(x)g(x)| \le \frac{f(x)^p}{p} + \frac{g(x)^q}{q}. \tag{1}$$

This implies that $|fg| \in L^1$ and

$$\int |fg|\, dx \le \frac{1}{p} \int |f|^p\, dx + \frac{1}{q} \int |g|^q\, dx = 1. \tag{2}$$

If $\|f\|_p = 0$ or $\|g\|_q = 0$, then $f = 0$ a.e. or $g = 0$ a.e. and thus $fg = 0$ a.e. In this case the inequality is an equality. If $\|f\|_p \ne 0$ and $\|g\|_q \ne 0$, then (2) holds for $\frac{f}{\|f\|_p}$ and $\frac{g}{\|g\|_q}$ so that we have

$$\int \frac{|fg|}{\|f\|_p \|g\|_q}\, dx \le 1,$$

or

$$\int |fg|\, dx \le \|f\|_p \|g\|_q.$$

Equality holds if and only if equality holds in (2) for $\frac{f}{\|f\|_p}$ and $\frac{g}{\|g\|_q}$ if and only if equality holds in (1) for $\frac{f}{\|f\|_p}$ and $\frac{g}{\|g\|_q}$ which by the equality case of Young's inequality holds if and only if

$$\frac{|f(x)|^p}{\|f\|_p^p} = \frac{|g(x)|^q}{\|g\|_q^q}$$

a.e. □

Theorem 6. *(Minkowski's Inequality) Let $1 \le p < \infty$ and $f, g \in L^p$. Then $f + g \in L^p$ and $\|f+g\|_p \le \|f\|_p + \|g\|_p$.*

Proof. If $p = 1$, then we have already proved the statement. Let $1 < p < \infty$. Then $q = \frac{p}{p-1}$, so $(p-1)q = p$. We have already shown that $f + g \in L^p$. Now

$$|f+g|^p = |f+g||f+g|^{p-1} \le |f||f+g|^{p-1} + |g||f+g|^{p-1}.$$

Observe that $|f+g|^{p-1} \in L^q$, since $(p-1)q = p$, and $\||f+g|^{p-1}\|_q = \|f+g\|_p^{\frac{p}{q}}$. Now Hölder's inequality implies

$$\int |f+g|\, dx \le \|f\|_p \|f+g\|_p^{\frac{p}{q}} + \|g\|_p \|f+g\|_p^{\frac{p}{q}} = (\|f\|_p + \|g\|_p)\|f+g\|_p^{\frac{p}{q}} \tag{3}$$

If $\|f+g\|_p = 0$ then the triangle inequality is obvious, otherwise we can divide both sides of (3) by $\|f+g\|_p^{\frac{p}{q}}$ to get

$$\|f+g\|_p^{p-\frac{p}{q}} \leq \|f\|_p + \|g\|_p,$$

which is the desired inequality, since $p - \frac{p}{q} = p(1-\frac{1}{q}) = 1$. $\square$

The above theorem shows that L^p for $1 \leq p < \infty$ is a normed space. Before proving that L^p is a Banach space, we recall that if in a metric space a Cauchy sequence has a convergent subsequence, then the Cauchy sequence converges.

Theorem 7. *(Riesz-Fisher) The space L^p for $1 \leq p < \infty$ is a Banach space.*

Proof. Let $\{f_n\}$ be a norm Cauchy sequence in L^p. Then we can find a subsequence $\{f_{n_k}\}$ such that $\|f_{n_k} - f_{n_{k+1}}\|_p < \frac{1}{2^k}$. By the above remark it suffices to show that $\{f_{n_k}\}$ converges in norm to a function $f \in L^p$. Define

$$g_m(x) = |f_{n_1}(x)| + \sum_{k=1}^{m} |f_{n_{k+1}}(x) - f_{n_k}(x)|.$$

Then g_m is measurable, $0 \leq g_1 \leq g_2 \leq \cdots$ and

$$\|g_m\|_p \leq \|f_{n_1}\|_p + \sum_{k=2}^{m} \|f_{n_{k+1}} - f_{n_k}\|_p \leq \|f_{n_1}\|_p + 1 = M.$$

Hence $0 \leq g_1^p \leq g_2^p \leq \cdots$ and $\int g_m^p \, dx \leq M^p$ for all m. Let $g(x) = \lim_{m\to\infty} g_m(x)$. Then g is measurable and by the Monotone Convergence Theorem $\int g^p \, dx = \lim_{m\to\infty} \int g_m^p \, dx \leq M^p < \infty$. Hence $g \in L^p$ and in particular $g(x) < \infty$ a.e.. If $g(x) < \infty$, then the telescoping series $f_{n_1}(x) + \sum_{k=1}^{\infty}(f_{n_{k+1}}(x) - f_{n_k}(x))$ converges absolutely, so $\lim_{k\to\infty} f_{n_k}(x)$ exists. Define therefore $f(x) = \lim_{k\to\infty} f_{n_k}(x)$ when $g(x) < \infty$ and $f(x) = 0$ elsewhere. Then $f_{n_k}(x) \to f(x)$ a.e., so f is measurable. Moreover $|f_{n_k}(x)| \leq g_{k-1}(x) \leq g(x)$ implies that $|f| \leq g$. This shows that $\int |f|^p \, dx \leq \int g^p \, dx < \infty$, so $f \in L^p$. It remains to show that f_{n_k} converges in norm to f. Observe first that $|f_{n_k} - f|^p \leq (|f_{n_k}| + |f|)^p \leq 2^p |g|^p$ and $|f_{n_k}(x) - f(x)|^p \to 0$ a.e. Hence by the Dominated Convergence Theorem $\int |f_{n_k} - f|^p \, dx \to 0$. i.e., $\|f_{n_k} - f\|_p \to 0$. $\square$

Corollary 8. *If $\{f_n\}$ converges to f in L^p, $1 \leq p < \infty$, then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k}(x) \to f(x)$ a.e.*

Proof. Every convergent sequence is a Cauchy sequence,and by the above proof every Cauchy sequence contains an a.e. convergent subsequence with limit equal to the norm limit of the Cauchy sequence.

$\square$

Recall that a subset A of metric space is **dense** in X, if $\overline{A} = X$. If X has a countable dense subset, then X is called **separable**.

Theorem 9. *The following sets of functions are dense in $L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$ and μ equall the Lebesgue measure.*

(i) *The simple functions.*
(ii) *The step functions.*
(iii) *The continuous functions with compact support.*

Proof. It suffices to show that every non-negative $f \in L^p$ is a norm limit of a sequence of functions from each of the sets.

For (i) let $\{\phi_n\}$ be a sequence of simple functions such that $0 \leq \phi_1 \leq \phi_2 \leq \cdots \uparrow f$. Then $|f - \phi_n|^p \leq 2^p |f|^p$ and $|f(x) - \phi_n(x)|^p \to 0$. By the Dominated Convergence Theorem we have $\int |f - \phi_n|^p \, dx \to 0$, i.e., $\|f - \phi_n\|_p \to 0$.

To prove (ii) it suffices by (i) to approximate a non-negative simple function and thus it suffices to approximate χ_E for a measurable set E with $m(E) < \infty$. Let $\epsilon > 0$. Then there exist almost disjoint rectangles $\{R_i\}_{i=1}^n$ such that $m(E \triangle \cup_i R_i) < \epsilon^p$. Then we have that $\|\chi_E - \sum_{i=1}^n \chi_{R_i}\|_p < \epsilon$.

To prove (iii) it suffices by (ii) to approximate the characteristic function of a rectangle by continuous functions with compact support. In the one dimensional case we have $f = \chi_{[a,b]}$. Then we can define g as the piecewise linear continuous function such that $g(x) = 0$ for $x \leq a - \frac{\epsilon^p}{2}$, $x \geq b + \frac{\epsilon^p}{2}$, and $g(x) = f(x)$ on $[a, b]$. Then $\|f - g\|_p < \epsilon$. In the d-dimensional case we can take for g the product of d such piecewise linear continuous functions. □

Corollary 10. *The Banach space $L^p(\mathbb{R}^d)$ is separable for $1 \leq p < \infty$.*

Proof. It is straightforward to check that the step functions with rational values and supported on rectangles with rational vertices is dense in the collection of all step functions, and therefore by (ii) above dense in L^p.

□