2.2 Projective Resolutions

Def: An object P in an abelian category $\mathcal{A}$ is **projective** if it satisfies the following: for each surjection $g: B \to C \to 0$, $\gamma: P \to C$, we have

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists\beta"'] \arrow[d, "\gamma"] & \\ B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

Rmk: 1. β may not be unique.

2. P is projective $\Leftrightarrow$ $\mathrm{Hom}(P, -)$ is right exact.

Prop: (2.2.1) An R-module is projective $\Leftrightarrow$ it's a direct summand of a free R-module.

Pf: $\Rightarrow$:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\iota"'] \arrow[d, "id"] & \\ F \arrow[r, "\pi"] & P \arrow[r] & 0 \end{tikzcd}$$

so P is a summand of F

$\Leftarrow$: Suppose $F = P \oplus Q$.

$$\begin{tikzcd} & P \oplus Q \arrow[d, shift left, "\pi"] \\ \tilde{F} \arrow[ur, dashed] \arrow[dr, dashed, "f\iota"'] & P \arrow[u, shift left, "\iota"] \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \arrow[r] & 0 \end{tikzcd}$$

$(f = f\pi)$

easy to check: $f = g\tilde{F}\iota$

E.g.: (2.2.2) Over some rings (like $\mathbb{Z}$, k, division rings), projective $\Rightarrow$ free.

However, it's not always true.

1. If $R = R_1 \oplus R_2$, then $P = R_1$, $Q = R_2$ (summands) are projective, but they are not free, since $(0,1)\cdot P = 0$.

2. Let $R = M_n(k)$, $V = k^n$, then $\dim_k R = n^2$, $\dim_k V = n$.

Moreover, V is an R-module, where $R = V^{\oplus n}$.

Thus V is R-projective.

Claim: V is not R-free. If not, ${}_RV \cong {}_RR^d$ (also ${}_kV \cong {}_kR^d$)

So $\dim_k V = d\cdot n^2$, but $\dim_k V = n$, a contradiction.

Rmk: The category $\mathcal{A}$ of finite abelian groups is an example of an abelian category with no projective objects. (no free objects)

We say an abelian category $\mathcal{A}$ has **enough projectives** if for every object A of $\mathcal{A}$, there is a surjection $P \to A$ with P projective.

Def: A chain complex P with each P_n projective is called a **chain complex of projectives**. Note: P may not be a projective object of $Ch(\mathcal{A})$.

E.X: (2.2.1) A chain complex P is a projective object in Ch iff it's a split exact complex of projectives.

Pf: $\Rightarrow$: claim 1: every P_n is projective.

Pf: Consider

$$\begin{array}{ccccccccc} \to & 0 & \to & B_n & \to & 0 & \to & \cdots \\ & & & \downarrow & & & & \\ \to & 0 & \to & C_n & \to & 0 & \to & \cdots \\ & & & \downarrow & & & & \\ \to & 0 & \to & 0 & \to & 0 & \to & \cdots \end{array}$$

so P_n is projective

claim 2: $P_\cdot$ is split exact.

Pf: Consider the surjection $\mathrm{cone}(P_\cdot) \to P[-1] \to 0$

$P_\cdot$ is projective, so is $P[-1]$.

$$\begin{tikzcd} & & P[-1] \arrow[dl, dashed, "i"'] \arrow[d, "id"] & \\ 0 \arrow[r] & P \arrow[r] & \mathrm{cone}(P_\cdot) \arrow[r, "\pi"'] & P[-1] \arrow[r] & 0 \end{tikzcd}$$

so $\mathrm{cone}(P_\cdot) \cong P_\cdot \oplus P[-1]$

$\mathrm{cone}(P_\cdot)$ is always split exact, so are $P_\cdot$ and $P[-1]$.

$\Leftarrow$: $P_\cdot$ is split exact, so $P_n \cong B_n \oplus B_{n-1}$, and B_n are projective

Let $P(n)$ be the chain complex: $\cdots \to 0 \to B_{n-1} \to B_{n-1} \to 0 \to \cdots$

so $P_\cdot \cong \bigoplus_{n\in\mathbb{Z}} P(n)$.

Consider the maps

$$\begin{tikzcd} & P_\cdot \arrow[d, "f=\oplus f_n"'] \arrow[dl, dashed, "\tilde{f}=\oplus \tilde{f}_n"'] \\ X_\cdot \arrow[r, "g"'] & Y_\cdot \arrow[r] & 0 \end{tikzcd}$$

where $f = \oplus f_n$

For each n, $P(n)$ is a split exact chain complex of projectives, so projective, thus we get $\tilde{f}_n : P(n) \to X_\cdot$.

Define $\tilde{f} = \oplus \tilde{f}_n$, we get $g\tilde{f} = f$.

E.x (2.2.2) If $\mathcal{A}$ has enough projectives, then so does $Ch(\mathcal{A})$.

Pf:

$$\begin{tikzcd} P_{n+1} \arrow[r, dashed, "d_{n+1}"] \arrow[d, two heads] \arrow[dr] & P_n \arrow[r, dashed, "d_n"] \arrow[d, two heads, "f_n"] & P_{n-1} \arrow[d, two heads] \\ B_{n+1} \arrow[r] & B_n \arrow[r] & B_{n-1} \end{tikzcd}$$

First, we get $d_n : P_n \to P_{n-1}$.

Second, we want to get a split exact complex from $\cdots \to P_{n+1} \to P_n \to P_{n-1} \to \cdots$

Indeed, $\mathrm{cone}(P_\cdot)[1]$ works.

$$\begin{tikzcd} P_{n-1} \oplus P_n \arrow[r] \arrow[d, "(-1)^{n-1}"] & P_{n-2} \oplus P_{n-1} \arrow[d, "(-1)^n"] \\ B_{n-1} \arrow[r] & B_{n-1} \end{tikzcd}$$

$(x, y) \mapsto x + dy$, $(-dx, dy - x) \mapsto dx$. $\square$

Def: (2.2.4) Let M be an object of $\mathcal{A}$. A left resolution of M is an exact complex $\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \to 0$. It's a projective resolution if each P_i is projective.

Lemma: (2.2.5) If an Abelian category $\mathcal{A}$ has enough projectives, then every object M in $\mathcal{A}$ has a projective resolution.

Pf:

$$\begin{tikzcd} \cdots \arrow[r] & P_1 \arrow[rr, "d_1 = \pi_1 \tilde{d}_1"] \arrow[dr, "\tilde{d}_1"'] & & P_0 \arrow[r, "\varepsilon"] & M \arrow[r] & 0 \\ & & \ker\varepsilon \arrow[ur, "\pi_1"'] & & & \end{tikzcd}$$

E.x (2.2.3) If $P_\cdot$ is a chain complex of projectives with $P_i = 0$ for $i < 0$, then a map $\varepsilon : P_0 \to M$ giving a resolution for M is the same thing as a quasi-isomorphism $\varepsilon : P_\cdot \to M_\cdot$, where $M_\cdot$ is the complex concentrated in degree 0.

Pf: $\to P_n \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon_0} M \to 0$ is a projective resolution is equivalent to say $H_i = 0$ for $i \geq 0$, and $P_0/\mathrm{Im}\, d_1 \cong M$, which is equivalent to say the following chain map

$$\begin{tikzcd} \to P_{n+1} \arrow[r] & P_n \arrow[r] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] \arrow[d] & 0 \arrow[r] & \cdots \\ & & & \cdots \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

is a quasi-isomorphism.

Thm: (2.2.6) (Comparison Theorem) Let $P_\cdot \xrightarrow{\varepsilon} M$ be a projective resolution of M and $f' : M \to N$ a map in $\mathcal{A}$. Then for every resolution $Q_\cdot \xrightarrow{\eta} N$, there is a chain map $f : P_\cdot \to Q_\cdot$ such that $\eta f_0 = f'\varepsilon$. The chain map is unique up to chain homotopy equivalence.

Pf: 1.

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "f_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1"] \arrow[dr, "f_0 d_1"] & P_0 \arrow[r] \arrow[d, "f_0"] \arrow[dr, "f'\varepsilon"] & M \arrow[r] \arrow[d, "f'"] & 0 \\ \cdots \arrow[r] & Q_2 \arrow[r] & \mathrm{Im} f_1 \arrow[r] & Q_1 \arrow[r, two heads, "\delta_1"'] & \mathrm{Im} f_0 \arrow[r, tail] & Q_0 \arrow[r, "\eta"'] & N \arrow[r] & 0 \end{tikzcd}$$

2. Now suppose we have f and g satisfying the above conditions.

$\eta f_0 = f'\varepsilon = \eta g_0 \Rightarrow \eta(f_0 - g_0) = 0 \Rightarrow \mathrm{Im}(f_0 - g_0) \subseteq \ker\eta = \mathrm{Im}\,\delta_1$

So we have

$$\begin{tikzcd} & P_0 \arrow[d, "f_0 - g_0"] \arrow[dl, dashed, "s_0"'] & \\ Q_1 \arrow[r, two heads, "\delta_1"'] & \mathrm{Im}\,\delta_1 \arrow[r] & 0 \end{tikzcd}$$

Similarly, get s_n.

Lemma: (2.2.8) (Horseshoe Lemma) Suppose we have the following projective resolutions:

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
\cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & A' \arrow[r] \arrow[d] & 0 \\
 & & & & A \arrow[d] & \\
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & A'' \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}$$

Then $P_\bullet \oplus Q_\bullet$ is a projective resolution of A.

Pf:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & P_0 \arrow[r, "\varepsilon"] \arrow[d] & A' \arrow[r] \arrow[d, "\iota"] & 0 \\
 & P_0 \oplus Q_0 \arrow[r, "d_0"] \arrow[d] & A \arrow[r] \arrow[d, "\pi"] & 0 \\
 & Q_0 \arrow[r, "\eta"'] \arrow[d] \arrow[ur, dashed, "\xi"] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

$P_i \oplus Q_i$ is projective since it's a summand of a free module

First, let's define $d_0 : P_0 \oplus Q_0 \to A$

$\pi : A \to A''$ is surjective, Q_0 is projective $\Rightarrow \exists\, \eta : Q_0 \to A$.

Define $d_0 : P_0 \oplus Q_0 \to A$ as $d_0 = \iota\varepsilon \oplus \eta$.

Easy to check d_0 is surjective, and the commutativity.

Now, we can get the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & P_1 \arrow[r] \arrow[d] & \ker\varepsilon \arrow[r] \arrow[d] & P_0 \arrow[r, "\varepsilon"] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\
 & P_1 \oplus Q_1 \arrow[r, "d_1"] \arrow[d] \arrow[ur, dashed] & \ker d_0 \arrow[r] \arrow[d] & P_0 \oplus Q_0 \arrow[r, "d_0"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & Q_1 \arrow[r] \arrow[d] & \ker\eta \arrow[r] \arrow[d] & Q_0 \arrow[r, "\eta"'] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

Haven't got it yet

Similarly, we get d_1, and d_1 is surjective. i.e., "$P_\bullet \oplus Q_\bullet$" is exact at $P_0 \oplus Q_0$. Repeat this, we get a projective resolution $P_\bullet \oplus Q_\bullet$ of A.

2.3 Injective Resolutions

Def: An object I in an abelian category $\mathcal{A}$ is *injective* if it satisfies the following universal lifting property: Given an injection $f : A \to B$ and a map $\partial : A \to I$, we have

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\partial"'] & B \arrow[dl, dashed, "\exists\, \tilde{\partial}"] \\
 & I &
\end{tikzcd}$$

We say $\mathcal{A}$ has enough injectives if for every object A in $\mathcal{A}$, there is an injection $A \to I$ with I injective.

Prop: If $\{I_\alpha\}$ is a family of injectives, then $\prod_\alpha I_\alpha$ is also injective.

Pf:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[dl, dashed] \\
 & \prod I_\alpha \arrow[d] & \\
 & I_\alpha &
\end{tikzcd}$$

Thm: (2.3.1) (Baer's Criterion) A right R-module E is injective iff for every right ideal J of R, every map $J \to E$ extends to a map $R \to E$.

Pf: $\Rightarrow$: Obvious

$\Leftarrow$: Suppose we have

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & B \\ & E & \end{tikzcd}$$

Let $\Sigma = \{$ extensions $\alpha' : A' \to E$ of $\alpha : A \to E \}$

Define the order $\alpha' : A' \to E \geq \alpha'' : A'' \to E$ as

$A' \supseteq A''$ and $\alpha'|_{A''} = \alpha''$

Then easy to check Σ has a maximal element $\beta : C \to E$.

Claim: $C = B$, so we get $\beta : B \to E$, thus E is injective.

If $C \neq B$, then $\exists\, b \in B$, but $b \notin C$.

Let $J = (C :_R b)$ be an ideal of R, and we have

a map $f : J \longrightarrow E$

$r \longmapsto \beta(rb)$

and this map extends to $\tilde{f} : R \longrightarrow E$.

Now, let $C' = C + Rb \supsetneq C$, and define

$F : C' \longrightarrow E$ which is a morphism.

$c + rb \longmapsto \beta(c) + \tilde{f}(r)$.

This contradicts with C being maximal.

Hence $C = B$ $\square$

Def: An abelian group M is called divisible if for $\forall x \in M$, $n \in \mathbb{Z}$, $\exists\, y \in M$ s.t $x = ny$.

Cor: (2.3.2) An $\mathbb{Z}$-module M (more generally, a PID-module M) is injective $\Leftrightarrow$ M is divisible.

Pf: ✓ Consider $0 \to (n) \to \mathbb{Z}$

Lemma: Every abelian group G can be embedded into a divisible group.

Pf: If G is free, then $G = \bigoplus_i \mathbb{Z}$, Let $D = \bigoplus_i \mathbb{Q}$,

then D is divisible and $0 \to G \to D$.

If G is not free, then we have $0 \to K \to F \to G \to 0$

so $G = F/K \hookrightarrow D/K$, where D is divisible.

D is divisible $\Rightarrow$ D/K is divisible.

Lemma: Let D be an $\mathbb{Z}$-injective module, R a ring, then $\mathrm{Hom}_{\mathbb{Z}}(R, D)$ is an R-injective module.

Pf:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[dr, "\varphi"'] & R \arrow[d, dashed, "\tilde{\varphi}"] \\ & & \mathrm{Hom}_{\mathbb{Z}}(R, D) \end{tikzcd} \Longrightarrow \begin{tikzcd} 0 \arrow[r] & {}_{\mathbb{Z}}I \arrow[r] \arrow[dr, "\psi"'] & {}_{\mathbb{Z}}R \arrow[d, dashed, "\tilde{\psi}"] \\ & & D \end{tikzcd}$$

$\varphi : 1 \mapsto x$

$\tilde{\varphi}_1 : 1 \mapsto y$

$x \mapsto y$

Thm: R-mod has enough injectives, i.e., every R-module can be embedded into an injective R-module.

Pf: $M \xhookrightarrow{\sim} \mathrm{Hom}_R(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, D)$

Rmk: The above theorem is the only one that we can not get from dualizing the projective result.

Def (2.3.9): A pair of functors $L: \mathcal{A} \to \mathcal{B}$ and $R: \mathcal{B} \to \mathcal{A}$ are **adjoint** if there is a natural bijection for all A in $\mathcal{A}$ and B in $\mathcal{B}$:

$$\tau_{AB}: \mathrm{Hom}_{\mathcal{A}}(A, R(B)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{B}}(L(A), B)$$

Here, "natural" means for $f: A \to A'$ and $g: B \to B'$:

$$\begin{tikzcd}
\mathrm{Hom}(A', R(B)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}(A, R(B)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}(A, R(B')) \arrow[d] \\
\mathrm{Hom}(L(A'), B) \arrow[r] & \mathrm{Hom}(L(A), B) \arrow[r] & \mathrm{Hom}(L(A), B')
\end{tikzcd}$$

We call L the **left adjoint** and R the **right adjoint**

E.g.:
- Forgetful functor: R-mod $\to$ Ab is left adjoint
- $\mathrm{Hom}(R, -)$: Ab $\to$ R-mod is right adjoint

Prop: Suppose $\mathcal{A}, \mathcal{B}$ are abelian categories. (L, R) is an adjoint pair, $L: \mathcal{A} \to \mathcal{B}$ is exact, and I is an injective object in $\mathcal{B}$. Then $R(I)$ is an injective object in $\mathcal{A}$.

Pf: It suffices to prove $\mathrm{Hom}_{\mathcal{A}}(-, R(I))$ is right exact.

$$0 \to A \to A'$$
$$\Rightarrow 0 \to L(A) \to L(A')$$
$$\Rightarrow \mathrm{Hom}(L(A'), I) \to \mathrm{Hom}(L(A), I) \to 0$$
$$\Rightarrow \mathrm{Hom}(A', R(I)) \to \mathrm{Hom}(A, R(I)) \to 0 \quad \square$$