

3.3

Ext for Nice Rings

SAPW

→ first we review a few things dealing with injective modules

Boer's criterion: (for injectives) An R -module E is injective iff

every R -module hom $\phi: I \rightarrow E$ when I is an ideal of R , extends to hom

$$R \rightarrow E$$

Pf: (\Rightarrow) follows from definition (\Leftarrow) let $N \in \mathcal{N}$ (R -mod) $\phi: N \rightarrow E$ (E has desired property)
 we need to show ϕ extends to a hom $N' \rightarrow E$. by Zorn's \exists max'l N' . s.t.
 $N \subseteq N' \subseteq N$ w/ des ϕ extension $\phi': N' \rightarrow E$. If $N' \neq N$, take an element $n \in N - N'$
 $\xrightarrow{\text{translates } n \text{ into } \phi(n)}$ and consider the ideal $I = N + (n)$ by hypothesis, we composite $\phi \xrightarrow{\text{in }} N' \xrightarrow{\phi'} E$
 extends to $I \rightarrow E$. Define $\phi'': N + I \rightarrow E$ by $\phi''(n+rn) = \phi'(n) + \phi(r)$
 now ϕ contradicts maximality of ϕ' , so we know $N = N'$.

→ Recall: when R is an integral domain. An R -module M is divisible if $\forall r \in R, r \neq 0$ for every nonzero element $x \in M$ (note this implies M is surjective)



Lemma: Let R be a P.I.D., then an R -module is injective iff it is divisible.

prf: (\Rightarrow) let E be injective, let $y \in E$ and $r \in R$ consider hom $\phi: r\mathbb{Z} \rightarrow E, x \mapsto y$
 then by Boer's exist $\phi': \mathbb{Z} \rightarrow E$ such exists ϕ , such $\phi' \xrightarrow{\text{in }} x \mapsto ry$ for $r \in \mathbb{Z}$ in $r\mathbb{Z} = E$
 (note this has been well defined since R is a domain)

(\Leftarrow) let I be an ideal of R , then $\exists r \in R$ s.t. $I = \langle r \rangle$, and let $\phi: I \rightarrow E$ (E is divisible)
 denote $x \in I$ as $x = rx$ then since E is divisible $y \in E$ s.t. $ry = x$ in $\phi: I \rightarrow E$
 b. $\phi(x) = \phi(rx) = r\phi(x) = ry = y$ so ϕ is surjective by Boer's we get E is injective

Corollary: for a \mathbb{Z} -mod B not a field we see that $0 \rightarrow B \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/B \rightarrow 0$ is exact
 \mathbb{Z}/B is 'injective'.

prf: but remember \mathbb{Z} is divisible when $I = \mathbb{Z}$ we must only show \mathbb{Z}/B is divisible
 w/ $I = \langle y \rangle \subset \mathbb{Z}/B$, let $r \in R$ we must also show $\exists x \in \mathbb{Z}/B$ s.t. $r[x] = [y]$, and not
 $\exists x \in \mathbb{Z}/B$ s.t. $r[x] \neq [y]$

Lemma 3.3.1: $\text{Ext}_{\mathbb{Z}}^n(A, B) = 0$ for $n \geq 2$ and all abelian groups A, B

(SAS
RU)

PF: map b into an injective I^0 , by writing the quotient $(I^0)^*$ is injective

↓, since, $\text{Ext}^n(A, B)$ is the cokernel of

(Recall $\text{Hom}(I, -)$ is
counit)

$$0 \rightarrow \text{Hom}(A, I^0) + \text{Hom}(A, I^1) \rightarrow 0$$

□

Calculation 3.3.2: Let B be a \mathbb{Z} -module, let's calculate $\text{Ext}_{\mathbb{Z}}^n(\mathbb{Z}/p\mathbb{Z}, B)$

Proof: we can act this by taking any projective resolution of $\mathbb{Z}/p\mathbb{Z}$

so we take

$$0 \rightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow 0 \quad (\text{as } \text{Res} \text{ } \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, B) \cong B, \text{ where do you take } \mathbb{Z}?)$$

then $\text{Hom}(-, B)$ is commutative

$$0 \rightarrow \text{Hom}(\mathbb{Z}, B) \xrightarrow{(\cdot p)} \text{Hom}(\mathbb{Z}, B) \rightarrow 0$$

$$0 \rightarrow B \xrightarrow{p} B \rightarrow 0$$

$$\text{then } \text{Ext}_{\mathbb{Z}}^n(\mathbb{Z}/p\mathbb{Z}, B) = \begin{cases} pB \leftarrow p-\text{torsion} & n=0 \\ B/pB & n=1 \\ 0 & n \geq 2 \text{ and lower values} \end{cases}$$

can see it!

Note: since \mathbb{Z} is projective, $\text{Ext}_{\mathbb{Z}}^n(\mathbb{Z}, B) = 0$, then for even for gen. abelian group

$A \cong \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/p_n\mathbb{Z}$ by taking a finite direct sum of $\text{Ext}^n(\mathbb{Z}/p\mathbb{Z}, B)$ groups

⇒ fin. gen. group in calculation is more complicated than in Tor

Example 3.3.3: let A be a torsion group, and write $A^0 = \text{Hom}(A, \mathbb{Q}/\mathbb{Z})$

we would like to now calculate $\text{Ext}_{\mathbb{Z}}^n(A, \mathbb{Z})$

consider the injective resolution $0 \rightarrow \mathbb{Z} \xrightarrow{\epsilon} \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \rightarrow 0$ (injective since \mathbb{Q} is divisible)

so we have cokernel of $0 \rightarrow \text{Hom}(A, \mathbb{Q}) \xrightarrow{\text{no a point wise}} \text{Hom}(A, \mathbb{Q}/\mathbb{Z}) \rightarrow 0$

so $\text{Ext}_{\mathbb{Z}}^0(A, \mathbb{Z}) = 0$ since A is torsion and hence every element has finite order

∴ any map would need to send $x \in A$ to $y \in \mathbb{Q}$ w/ finite order
∴ there only element of finite order is 0!

and hence $\text{Ext}_{\mathbb{Z}}^1(A, \mathbb{Z}) = A^0$

proposition 3.3.4: For all n and all rings R

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$$1. \text{Ext}_R^n\left(\bigoplus_{\alpha} A_{\alpha}, B\right) \cong \prod_{\alpha} \text{Ext}_R^n(A_{\alpha}, B)$$

$$2. \text{Ext}_R^n(A, \prod_B B_B) \cong \prod_B \text{Ext}_R^n(A, B_B)$$

FL:

If $P_{\alpha} \rightarrow A_{\alpha}$ are projective resolutions, then so is $\bigoplus P_{\alpha} \rightarrow \bigoplus A_{\alpha}$, since for I_p in facture and $\text{Tor}_p \rightarrow \prod I_p$. Since $\text{Hom}(\bigoplus A_{\alpha}, B) = \prod \text{Hom}(A_{\alpha}, B)$ and $\text{Hom}(A, \prod I_p) = \prod \text{Hom}(A, I_p)$ it then follows from the fact that for any functors C_R or cochain complexes

$$H^0(\pi C_R) \cong \prod H^0(C_R)$$

Examples 3.3.5:

1. If $p^2 \mid n$ and $A = \bigoplus^{\omega} \mathbb{Z}/p$ then by exercise 3.3.2 and prop 3.3.4 we get $\text{Ext}_{\mathbb{Z}/p}^n(A, \mathbb{Z}/p) \cong \prod^{\omega} \mathbb{Z}/p$ and hence a \mathbb{Z}/p -torsion class in \mathbb{Z}/p .

2. If B is a \mathbb{Z}/p -module $\prod_{p-\text{tors}} \mathbb{Z}/p$ then B is \mathbb{Z}/p -torsion and $\text{Ext}_{\mathbb{Z}/p}^1(A, B) = \prod_{p-\text{tors}} \mathbb{Z}/p$.

From calculation 3.3.2 and prop 3.3.4 we reach $\Rightarrow \text{Ext}_{\mathbb{Z}/p}^1(A, B) \cong \prod_{p-\text{tors}} \mathbb{Z}/p$ if B is divisible i.e. injective.

Lemma 3.3.6: Suppose that R is a commutative ring, so that $\text{Hom}(A, B)$ and the $\text{Ext}_R^n(A, B)$ are actually R -modules. If $\mu: A \rightarrow A$ and $\nu: B \rightarrow B$ are multipliers by $r, s \in R$, so are the induced mappings μ_r and ν_s of $\text{Ext}_R^n(A, B)$ for all n .

Proof: pick a projective res. $P \rightarrow A$. Mult. by r is a ring endomorphism $\tilde{\mu}: P \rightarrow P$ (it is commutative!) so map $\text{Hom}(P, B)$ or $\text{Hom}(A, B)$ is mult. by r , since it goes $\text{Ext}_R^n(P, B) \rightarrow \text{Ext}_R^n(A, B)$ for all n (takles $\text{pt } P_n \rightarrow \text{pt } (rP_n) = r\text{pt } P_n$), and the map H^0 on the subgroups $\text{Ext}_R^n(A, B)$ of $\text{Hom}(P, B)$ is also mult. by r . The argument for ν_s is similar, works as injective resolution $B \rightarrow I$. \square

Corollary 3.3.7: If R is commutative and A is actually an R/I -module, then for every R -module B the R -modules $\text{Ext}_R^n(A, B)$ are actually R/I -modules.

Lemma 3.3.8: If A a fin. generated R -mod. Then for every cont. mult. S in \mathbb{Z} , ϕ is an isomorphism

SAS PU

$$\phi: S^{-1}\text{Hom}_R(A, B) \cong \text{Hom}_{S^{-1}R}(S^{-1}A, S^{-1}B)$$

Proof: ϕ is trivially an iso when $A=R$, and since H is (finitely) additive, ϕ is also an isomorphism when $A=R^m$. The result now follows from the \mathbb{Z} -lemma and the following diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & S^{-1}\text{Hom}_R(A, B) & \rightarrow & S^{-1}\text{Hom}_R(F^n A, B) & \rightarrow & S^{-1}\text{Hom}_R(F^n B, B) \\ & & \downarrow \phi & & \cong \downarrow & & \cong \downarrow \\ 0 & \rightarrow & \text{Hom}_{S^{-1}R}(S^{-1}A, S^{-1}B) & \rightarrow & \text{Hom}_{S^{-1}R}(S^{-1}F^n A, S^{-1}B) & \rightarrow & \text{Hom}_{S^{-1}R}(S^{-1}F^n B, S^{-1}B) \end{array}$$

Recall the def'n of metrization, i.e. ideals are fin. gen. when R/I is fin. generated.

i.e. $R^n \xrightarrow{\exists!} R \xrightarrow{\exists!} R/I \rightarrow 0$, also since fin. gen. module (well-known) is a nat. suble

(i.e. no generator is fin. gen.) the following there is also fin. gen., and hence esp. fin. gen. R -mod, if A has a (non-triv.) $F \rightarrow A$ in which F is fin. gen. free R -mod.

Proposition 3.3.10: Let A be a fin. gen. module over a com. Noeth. ring R .

Then for every mult. set. S , all numbers b , and all n

$$\phi: S^{-1}\text{Ext}_R^n(A, B) \cong \text{Ext}_{S^{-1}R}^n(S^{-1}A, S^{-1}B)$$

Proof: Claim a (c.v. $F \rightarrow A$) by fin. gen. free R -mod. Then $S^{-1}F \rightarrow S^{-1}A$ is a m.d. fin. gen. for $S^{-1}R$ -mod

Since S^{-1} is a exact functor from R -mod to $S^{-1}R$ -mod (equivalently, for each $i \in \mathbb{Z}$) $\frac{i(x)}{S} = 0 \Rightarrow \exists i \cdot i(x) = 0$

$$S^{-1}\text{Ext}_R^0(A, B) = S^{-1}(\text{H}^0\text{Hom}_R(F, B)) \cong \text{H}^0(S^{-1}\text{Hom}_R(F, B)) \quad \text{since exact!}$$

$$\text{by prop. } \xrightarrow{\cong} \text{H}^0\text{Hom}_{S^{-1}R}(S^{-1}F, S^{-1}B) = \text{Ext}_{S^{-1}R}^0(S^{-1}A, S^{-1}B)$$

Corollary 3.3.11: (Localization of Ext) If R is commutative noeth. and A is fin. gen. R -mod

then the following are equiv. for all B and all n :

1. $\text{Ext}_R^n(A, B) = 0$
2. for every prime ideal $p \subset R$, $\text{Ext}_{R_p}^n(A_p, B_p) = 0$
3. for every maximal ideal \mathfrak{m} in R , $\text{Ext}_{R_{\mathfrak{m}}}^n(A_{\mathfrak{m}}, B_{\mathfrak{m}}) = 0$

3.4 Ext and Extending (by answering the question why Ext
like for sets useful to know
Ext tells us about extensions!)

Ext
RV

Defn: An **Extension** \mathcal{E} of A by B is an exact sequence $0 \rightarrow B \rightarrow X \rightarrow A \rightarrow 0$
Two extensions \mathcal{E} and \mathcal{E}' are **equivalent** if there is a commutative diagram

$$\mathcal{E}: 0 \rightarrow B \rightarrow X \rightarrow A \rightarrow 0 \quad \begin{matrix} \parallel & \downarrow \cong & \parallel \\ & & \end{matrix}$$

an extension is **split** if it is equivalent
to $0 \rightarrow B \xrightarrow{(0,1)} A \oplus B \rightarrow A \rightarrow 0$

$$\mathcal{E}' : 0 \rightarrow B \rightarrow X' \rightarrow A \rightarrow 0$$

Lemma 3.4.1: If $\text{Ext}'(A, B) = 0$, then every extension of A by B is split.

Pf: Let \mathcal{E} be the extension $0 \rightarrow B \rightarrow X \rightarrow A \rightarrow 0$ then we get the exact sequence
 $\text{Hom}(X, B) \xrightarrow{\text{ev}_B} \text{Hom}(B, B) \rightarrow \text{Ext}'(A, B)$.

So if $\text{Ext}'(A, B) = 0$ then $\text{Hom}(X, B) \xrightarrow{\text{ev}_B} \text{Hom}(B, B)$ is surjective and hence exists
 $\sigma: B \rightarrow X$ s.t. $\sigma \circ \text{ev}_B = \text{Id}_B$ and thus $0 \rightarrow B \rightarrow X \rightarrow A \rightarrow 0$ is split! \square

Proposition 3.4.2: (from the previous part of a proof) we see from this

that there all we need is to "lift" Id_A and since $\text{Hom}(A, x) \rightarrow \text{Hom}(A, x) / \text{Ext}'(A, B)$

this amounts to Id_B of $\text{ker}(\text{Hom}(B, B) \xrightarrow{\delta} \text{Ext}'(A, B))$ so we can consider the class

$\Theta(\mathcal{E}) = \delta(\text{Id}_B)$ in $\text{Ext}'(A, B)$ as some sort of **Obstruction** to \mathcal{E} to being split,

i.e. it's split iff $\Theta(\mathcal{E})$ is zero (i.e. in the kernel) or as the book says "vanishes".

Also notice that if X' and X are equivalent extensions then $\text{Hom}(X, B) \cong \text{Hom}(X', B)$

and, the naturality of δ $\Theta(\mathcal{E}) = \Theta(\mathcal{E}')$ and hence $\Theta(\mathcal{E})$ only depends on the equivalence class of \mathcal{E}

this is made more precise in the following Theorem, where we can think $\Theta(\mathcal{E})$ as a "code"

of what to send Id_A in $\text{Ext}'(A, B)$

$$\begin{array}{ccc} \mathcal{E}: 0 \rightarrow B \rightarrow X \rightarrow A \rightarrow 0 & \xrightarrow{\text{ev}_B: \text{Hom}(A, B) \rightarrow \text{Hom}(B, B)} & \text{Ext}'(A, B) \\ \parallel \cong \downarrow \parallel & \Rightarrow \parallel \cong \uparrow & \parallel \text{G} \uparrow \\ \mathcal{E}': 0 \rightarrow B \rightarrow X' \rightarrow A \rightarrow 0 & \xrightarrow{\text{ev}_{X'}: \text{Hom}(A, X') \rightarrow \text{Hom}(X', B)} & \text{Ext}'(A, B) \end{array}$$

Theorem 3.4.3: Given two R -modules A and B , the mapping $\Theta: \mathcal{E} \mapsto \delta(\text{Id}_B)$
establishes a 1-1 correspondence

$$\left\{ \begin{array}{c} \text{equiv. classes of } \mathcal{E} \\ \text{extensions of } A \\ \text{by } B \end{array} \right\} \xleftrightarrow{1-1} \text{Ext}'(A, B)$$

in which the split extension corresponds to the element of $\text{Ext}'(A, B)$

Proof: Fix an exact seq $0 \rightarrow M \xrightarrow{f} P \rightarrow A \rightarrow 0$ w/ P projective (ex. via Ext^1 properties)

Apply $\text{Hom}(-, B)$ yields an exact sequence

$$\text{Hom}(P, B) \rightarrow \text{Hom}(M, B) \xrightarrow{\delta} \text{Ext}^1(A, B) \rightarrow 0 = \text{Ext}^1(P, B)$$

Since P is projective

Given $x \in \text{Ext}^1(A, B)$, choose $p \in \text{Hom}(M, B)$ w/ $\delta(B) = x$ (by surjectivity). Let X be the cokernel

of $M \rightarrow P \oplus B$, $m \mapsto (j(m), -f(m))$. Now we get a diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & M & \xrightarrow{f} & P & \xrightarrow{g} & A & \rightarrow & 0 \\ & & B & \downarrow & \downarrow \sigma & & \parallel & & \\ \text{g: } & 0 & \rightarrow & B & \rightarrow & X & \rightarrow & A & \rightarrow & 0 \\ & & & & i & & \sigma & & & \\ & & & & \text{(and } \sigma \text{ is } P \rightarrow P \oplus B \xrightarrow{g} X\text{)} & & & & \end{array}$$

To see that $\text{g: } 0 \rightarrow X$ is exact (and hence X is an extension of A by B) note that first, i is an inclusion $B \rightarrow P \oplus B \rightarrow X$ since if $b \in \ker(i) = \ker(0, 1) \in \text{Im}(M \rightarrow (j(m), f(m)))$ then $j(m) = 0$ and $f(m) = 0$ which implies $m = 0$. Next, $\text{Im}(i) = \text{Im}(j)$ since $j(m) = b$ implies $i(b) = j(m)$. Finally, $\text{Im}(i) \subseteq \ker(\sigma)$ since $\sigma(b) = f(b) = 0$ implies $f(b) = 0$ and $f(b) = g(j(b)) = g(b)$.

When the map $X \rightarrow A$ is induced by the map $f: B \rightarrow A$ and $P \rightarrow A$, it is injective. We will define an equivalence $[p, b] \equiv [p', b']$ for $p, p' \in P$ and $b, b' \in B$ if $b = b' + f(p) - f(p')$ and $p(b) = -(b - b')$. Action $\text{Ext}^1(A, B)$ is defined (take $\text{Ext}^1([p])$): $[0] \mapsto 0$ follows straightforwardly.

So for any Θ (repn't $P \rightarrow A$ and $m \in [p, b] \mapsto a$ for any $b \in B$ such that Θ is only defined if $[p, b] = 0$ w.r.t. Θ for one being primary) and any injection $\exists m \in M$ s.t. $j(m) = 0$ then $-f(m) = \text{coker } P \rightarrow A$! Finally we need to show $\ker(\theta \circ A) = \text{Im}(\theta) = \Theta$ where $\text{Im}(\theta) \subseteq \ker(\theta \circ A)$

so let $(p, b) \in \ker(\theta \circ A)$ then $\theta \circ A \circ (p, b) = 0 \Rightarrow \theta \circ A \circ (p, b) = 0 \Rightarrow \text{Im}(\theta) = \Theta$ since $\text{coker } P \rightarrow A = [(p, -f(m))] = 0$

Now $\text{coker } P \rightarrow A = \text{Ext}^1(A, B) \neq [0]$ so $\theta \circ A \circ (p, b) = 0$ implies $(p, b) \in \ker(\theta)$ i.e. $\exists m \in M$ s.t.

$j(m) = p - f(m) = b - b'$ (where b is a unique lift of b in B s.t. $j(m) = p - f(m)$ as $j(m) = p - f(m)$ is unique since $j(m) = p - f(m) \Rightarrow j(m) = p - f(m) \Rightarrow j(m) = p - f(m)$)

Now we need to use the injectivity to prove $\theta(\theta) = X$

$$\begin{array}{ccc} & & X \\ & \swarrow & \uparrow & \searrow \\ \text{A: } & \text{Hom}(H, B) & \rightarrow & \text{Ext}^1(A, B) \\ & \uparrow & \beta \uparrow & \parallel \\ & \text{Id}_B & \text{Hom}(B, B) & \rightarrow & \text{Ext}^1(A, B) \\ & & \text{Id}_B & \longrightarrow & X \end{array}$$

and this shows Θ is surjective! (note the choice of X was arbitrary!)

Now notice this procedure actually defines a set map from $\text{Ext}^1(A, B)$ to the set of equivalence classes of extensions. For if $\beta \in \text{Hom}(H, B)$ is another lift of θ then there is an $\ell \in \text{Hom}(P, B)$

such that $\beta = b + f_j$ (by exactness of θ) so if $X^1 = \text{coker}(m \mapsto (j(m), -(\beta + f_j)(m)))$

we claim β and β_j are equivalent to see the defining map $\Phi: X \rightarrow X^1$ as $\text{coker} \beta \xrightarrow{\Phi} [\beta, b + f_j]_{X^1}$ first well defined if $(p, b) \in [\beta, b]_X$ i.e. $\exists m \in M$ s.t. $j(m) = p - p'$ as $p(m) = b - b'$

then $[\beta, b + f_j]_{X^1} = [\beta, b + f(p)]_{X^1}$ since $j(m) = p - p'$ and $\beta'(m) = b - b' + f(p) - f(p') = \beta(m) + f_j(m)$ hence $f(p) = b - b'$

is injective as $[\beta, b + f_j]_{X^1} \mapsto [\beta, b + f(p)]_{X^1} \hookleftarrow [\beta, b]_X$ and hence $[\beta, b + f(p)]_{X^1} \equiv [\beta, b + f(p)]_X$

i.e. $\exists m \in M$ s.t. $j(m) = p - p'$ and $\beta'(m) = b - b' + f(p) - f(p') = \beta(m) + f_j(m)$ hence $f(p) = b - b'$

hence $[\beta, b]_X \equiv [\beta, b]_{X^1}$ to see surjective wrt $[\beta, b]_{X^1}, \ell \in X^1$ well just choose

$$[\beta, b + f(p)]_{X^1} \mapsto [\beta, b + f(p)]_{X^1} \cong [\beta, 0]_{X^1} \checkmark$$

well that's great now we want to show its an equivalence

SAS PV

$$\begin{array}{c} \mathfrak{g}: 0 \rightarrow B \xrightarrow{i} X \rightarrow A \rightarrow 0 \\ \parallel \quad \downarrow d \quad \parallel \\ \mathfrak{g}: 0 \rightarrow B \xrightarrow{i} X' \rightarrow A \rightarrow 0 \end{array}$$

i.e. all we need is the commutativity of this picture... (this is already established so.)
 So let $b \in B$ then $i(b) = [0, b]_X \xrightarrow{d} [0, i(b)]_{X'} = i'(b)$
 or $[P, b]_X \xrightarrow{g(p)} \leftarrow [P, b]_{X'}$ now commutes thus an equivalence

denote this new equivalence map as Ψ

Finally given an extension X' we can find a map $\tau: P \rightarrow X'$ to lift the properties of P (other objects)
 i.e. note $\begin{array}{ccc} & P & \\ \tau \downarrow & \nearrow & \\ X & \rightarrow & A \end{array}$ now all we need is $\tau: P \rightarrow B$ s.t.

$$\begin{array}{c} 0 \rightarrow M \xrightarrow{j} P \xrightarrow{g} A \rightarrow 0 \\ \downarrow \gamma \quad \downarrow \tau \quad \parallel \\ \mathfrak{g}: 0 \rightarrow B \xrightarrow{i} X \rightarrow A \rightarrow 0 \\ \qquad \qquad \qquad g' \end{array}$$

to find τ note that
 a quotient of a projective
 is projective since it
 is BT free in P/BQBT free!

$$\begin{array}{ccc} & M & \\ \tau \downarrow & \nearrow & \\ B & \rightarrow & \text{ker}(i \rightarrow A) \end{array}$$

now we claim that $X \cong \text{coker}(m \mapsto (j(m), -\gamma(m)))$ to see this note that

check $X \cong \text{coker}(m \mapsto (j(m), -\gamma(m)))$ to do this we build a map $\tau: X \rightarrow Y \Leftrightarrow [A, M] \mapsto \tau(p) + i(b)$

well defined: $[P, b] = [P', b] \Rightarrow \exists m \in M : j(m) - p - p' \rightarrow \gamma(m) = b - b'$ or in $\tau(p-p') - i(b-b') = \tau(j(m)) - i(\gamma(m)) = 0$
 by commutativity. to see injective assume $[P, b] \rightarrow \tau(p) + i(b) = \tau(p') + i(b') \leftarrow [P', b']$ then

$$\tau(p-p') + i(b-b') = 0 \Rightarrow \tau(p-p') = -i(b-b') \Rightarrow g(p-p') = g'(\tau(p-p')) = g'(i(b-b')) = 0 \quad (\text{since } g \text{ exact})$$

hence from this $j(m) = p-p' \Rightarrow i(b-b') = i(\gamma(m)) \Rightarrow b-b' = \gamma(m)$ (since γ injective) as desired

to see surjective if $x \in X$ then $\exists p \in P$ s.t. $g(p) = g(x)$ and then $x - \tau(p) \in \text{ker}(g')$

and hence $\exists b \in B$ s.t. $i(b) = x - \tau(p)$ so $x = \tau(p) + i(b)$ and hence $[P, b] \mapsto x$ as desired!

now $\Psi(\Theta(\mathfrak{g})) = \mathfrak{g}$ here Θ is bijective \square

Definition 3.4.4: (Baer Sum) Let $\xi: 0 \rightarrow B \rightarrow X \rightarrow A \rightarrow 0$ and $\xi': 0 \rightarrow B \rightarrow X' \rightarrow A \rightarrow 0$

be two extensions of A by B . Let X'' be the pullback $\{(x, x') \in X \times X' : \bar{x} = \bar{x}' \text{ in } A\}$

$$\begin{array}{ccc} X'' & \xrightarrow{\quad} & X' \\ \downarrow \gamma & & \downarrow \\ X & \xrightarrow{\quad} & A \end{array}$$

X'' contains 3 copies of B : 0×0 , $0 \times B$, and the skew diagonal $\{(b, 0) : b \in B\}$. The copies 0×0 and $0 \times B$ can be identified in the quotient Y of X'' by the skew diagonal (i.e., $x'' = 0$ or $x'' \in B$). Since $X''/(0 \times B) \cong X$ and $X''/B \cong A$ it is immediate that the sequence $d: 0 \rightarrow B \rightarrow Y \rightarrow A \rightarrow 0$

is also an extension of A by B . The ^{equivalence} class of d is called the **Baer sum** of the extensions ξ and ξ' , since the construction was introduced by L. Baer in 1934.

Corollary 3.4.5: The set of (equivalence classes of) extensions is an abelian group under

Baer sum, with zero being the class of split extensions. The map Θ is an isomorphism of abelian groups.

Proof: notice if we show that $\Theta(d) = \Theta(\xi) + \Theta(\xi')$ in $\text{Ext}^1(A, B)$. This will prove the Baer sum is well defined up to equivalence and the corollary will follow. (since Θ is injective!)

First by the properties of a pullback we get a unique map

$$\begin{array}{ccc} P & \xrightarrow{\quad} & X' \\ \gamma & \searrow & \downarrow \\ x & \downarrow & \downarrow \\ x & \xrightarrow{\quad} & A \end{array}$$

where γ and γ' are built like in the proof of the last theorem and define $\tilde{\gamma}: P \rightarrow Y$ as the map induced by the quotient, if we define $\delta: M \rightarrow B$ and $\delta': M \rightarrow B$ as in the previous proof again it is easy to see $\tilde{\gamma}$ is induced by $\gamma + \gamma'$ as in the last proof so the following commutes

$$0 \rightarrow M \rightarrow P \rightarrow A \rightarrow 0 \quad \text{and from the last diagram} \quad \Theta(d) = \delta(\gamma + \gamma') = \delta(\gamma) + \delta(\gamma') = \Theta(\xi) + \Theta(\xi')$$

$$\delta: 0 \rightarrow B \rightarrow Y \rightarrow A \rightarrow 0$$

□

↗ This shows why we call it ext

Since $\text{Ext}^1(A, B) \cong$ the abelian group of (equivalence classes) of the extensions of A by B

Vista 3.4.6: (Yoneda Ext groups) we can define $\text{Ext}^n(A, B)$ in any abelian category \mathcal{C} , even if it has no projectives and no injectives, to be the set of equivalence classes of \mathcal{C} -extensions under Baer sum (if indeed it is a set), the Freyd-Mitchell embedding then.

Shows that $\text{Ext}^n(A, B)$ is an abelian group - even though one could prove this fact independently. Similarly, we can recapture the groups $\text{Ext}^n(A, B)$ w/o mentioning projectives or injectives.

This approach is due to Yoneda. An element of the Yoneda $\text{Ext}^n(A, B)$ is an equivalence class of exact sequences of the form $\xi: 0 \rightarrow B \rightarrow x_n \rightarrow \dots \rightarrow x_1 \rightarrow A \rightarrow 0$

The equivalence relation is generated by the relation that $\xi' \sim \xi$ if there is a diagram

$$\xi: 0 \rightarrow B \rightarrow x_n \rightarrow \dots \rightarrow x_1 \rightarrow A \rightarrow 0$$

$$\xi': 0 \rightarrow B \rightarrow x'_n \rightarrow \dots \rightarrow x'_1 \rightarrow A \rightarrow 0$$

\Rightarrow "add" ξ and ξ' when $n \geq 2$, let x''_n be the pull back of x_1 and x'_1 over A , let x''_n be the pushout of x_n and x'_n under B , and let y_n be the quotient of x''_n by the Stein diagonal copy of B . Then $\xi + \xi'$ is the class of extensions

$$0 \rightarrow B \rightarrow y_n \rightarrow x_{n-1} \oplus x'_{n-1} \rightarrow \dots \rightarrow x_2 \oplus x'_2 \rightarrow x''_1 \rightarrow A \rightarrow 0$$

Now suppose that it has enough projectives. If $P \rightarrow A$ is a projective res., then composing the 2.2.6 yields a map from P to ξ hence a diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & M & \rightarrow & P_{n-1} & \rightarrow & \dots \rightarrow P_0 \rightarrow A \rightarrow 0 \\ & & \downarrow \beta & & \downarrow \gamma_n & & \downarrow \delta \\ \xi: 0 & \rightarrow & B & \rightarrow & x_n & \rightarrow & \dots \rightarrow x_1 \rightarrow A \rightarrow 0 \end{array}$$

by dimension shifting, there is an exact sequence:

$$\text{Hom}(P_n, B) \rightarrow \text{Hom}(P_{n-1}, B) \rightarrow \text{Ext}^n(A, B) \rightarrow 0$$

The relation $\Theta(\xi) = \beta(\beta)$ gives the 1-1 correspondence between the Yoneda Ext^n and the derived fun Ext^n .