The Limits that come up in Improper Integrals

Main Topic # 1: [‘Basic’ Limits] These are the first type of limits you probably did in Calc I and/or Pre-Calc

Commonly Occurring Limits

- $\lim_{x\to\infty} \frac{\ln(x)}{x} = 0$
- $\lim_{x\to\infty} c^x = 0 \quad (|c| < 1)$
- $\lim_{x\to\infty} c^{\frac{1}{x}} = 1 \quad (c > 0)$
- $\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}$

Problem 1. Determine the following limits. **DO NOT** simply write *undefined/DNE*, please indicate whether it ‘diverges to $\pm\infty$’ or ‘oscillates’ etc. That is please indicate the behavior of all the following limits.

i. $\lim_{x\to\infty} \ln(x)$

iv. $\lim_{x\to\infty} 4^{\frac{1}{x}}$

vii. $\lim_{x\to 0} \frac{\sin(2x)}{x}$

ii. $\lim_{x\to\infty} 4^x$

v. $\lim_{x\to\infty} \frac{1}{x}$

viii. $\lim_{x\to\infty} \frac{9x^4 - 7x^3 + 2x - 5}{22x - 3x^3 - x^4 + 12}$

iii. $\lim_{x\to -\infty} 4^x$

vi. $\lim_{x\to\infty} \sin(x)$

ix. $\lim_{x\to 1} \frac{x - 1}{\sqrt{x^2 + 2} - \sqrt{3x}}$

Main Topic # 2: [L'hopitals] This is L'hopitals rule basics

Indeterminate forms

Many times when working limits we want to write indeterminate forms like:

$$\frac{0}{0} \quad \frac{\pm\infty}{\pm\infty} \quad 0\cdot(\pm\infty) \quad 1^{\infty} \quad 0^0 \quad \infty^0 \quad \infty-\infty$$

L'Hopital's

Basics: Suppose that we have one of the following cases,

$$\lim_{x\to a}\frac{f(x)}{g(x)} \text{``}=\text{''} \frac{0}{0} \qquad \textbf{OR} \qquad \lim_{x\to a}\frac{f(x)}{g(x)} \text{``}=\text{''} \frac{\pm\infty}{\pm\infty}$$

where a can be any real number, infinity or negative infinity. In these cases we have

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

L'Hopital's for $0\cdot(\pm\infty)$

Suppose that we have,

$$\lim_{x\to a} f(x)\cdot g(x) \text{``}=\text{''}\ 0\cdot(\pm\infty)$$

Then we can write as either

$$f(x)\cdot g(x) = \frac{f(x)}{\left(\frac{1}{g(x)}\right)} \qquad \textbf{OR} \qquad f(x)\cdot g(x) = \frac{g(x)}{\left(\frac{1}{f(x)}\right)}$$

and next try to use L'Hopitals

L'Hopital's for other indeterminate forms

Suppose that we have,

$$\lim_{x\to a} f(x)^{g(x)} \text{``}=\text{''}\ 1^{\infty} \qquad \textbf{OR} \qquad \lim_{x\to a} f(x)^{g(x)} \text{``}=\text{''}\ 0^0 \qquad \textbf{OR} \qquad \lim_{x\to a} f(x)^{g(x)} \text{``}=\text{''}\ \infty^0$$

Then we can write

$$\ln\left(f(x)^{g(x)}\right) = g(x)\cdot\ln\left(f(x)\right) \qquad \textbf{AND} \qquad f(x)^{g(x)} = e^{\ln\left(f(x)^{g(x)}\right)}$$

AND SO

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{\ln\left(f(x)^{g(x)}\right)} = e^{\lim_{x\to a}\ln\left(f(x)^{g(x)}\right)}$$

Now notice that

$$\lim_{x\to a}\ln\left(f(x)^{g(x)}\right) = \lim_{x\to a} g(x)\cdot\ln\left(f(x)\right)$$

is now in one of the previous indeterminate forms!

Problem 2. Find the following limits using L'hopitals rule.

i. $\lim_{x\to 2} \dfrac{x^3 - 7x^2 + 10x}{x^2 + x - 6}$

ii. $\lim_{w\to -4} \dfrac{\sin(\pi w)}{w^2 - 16}$

iii. $\lim_{t\to\infty} \dfrac{\ln(3t)}{t^2}$

iv. $\lim_{z\to 0} \dfrac{\sin(2z) + 7z^2 - 2z}{z^2(z+1)^2}$

v. $\lim_{x\to -\infty} \dfrac{x^2}{\mathbf{e}^{1-x}}$

vi. $\lim_{z\to\infty} \dfrac{z^2 + \mathbf{e}^{4z}}{2z - \mathbf{e}^{z}}$

vii. $\lim_{t\to\infty} \left[t \ln\left(1 + \dfrac{3}{t}\right)\right]$

viii. $\lim_{w\to 0^+} \left[w^2 \ln\left(4w^2\right)\right]$

ix. $\lim_{x\to 1^+} \left[(x-1)\tan\left(\dfrac{\pi}{2}x\right)\right]$

x. $\lim_{y\to 0^+} \left[\cos(2y)\right]^{\frac{1}{y^2}}$

xi. $\lim_{x\to\infty} \left[\mathbf{e}^x + x\right]^{1/x}$