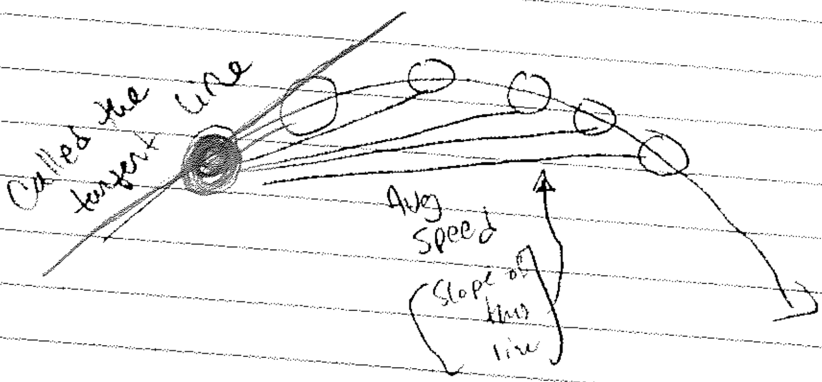
Instaneous Rate of Change / Derivative

<u>lets consider throwing a Ball:</u>



* play animation *

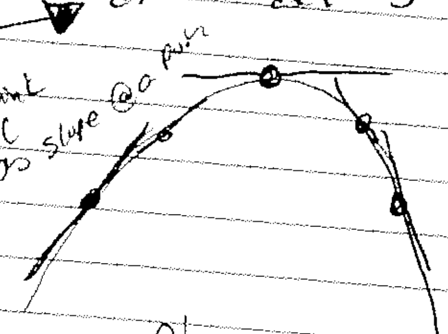
* call this the instaneous rate of change

The derivative function

denote: $f'(x)$ = instaneous rate of change of f @ x

notation: (if $y = f(x)$) $\frac{dy}{dx}$ or $\frac{d}{dx}[f(x)]$ or $f'(x)$

kinda looks like $\frac{\Delta y}{\Delta x}$



- $f' > 0$ f is increasing
- $f' < 0$ f is decreasing
- * $f' = 0$ neither * (important)

$\pm$ indicates direction

* think about speed + direction *

understand^ing the derivative:

The derivative is change!

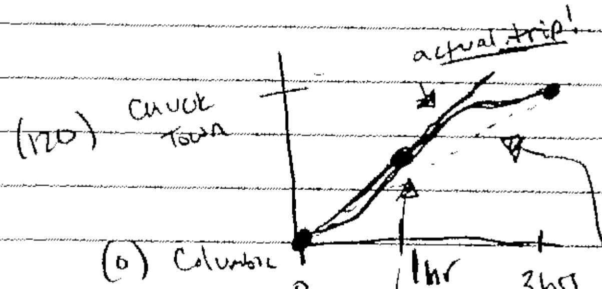
[in an instant]

like a polaroid!

example:

To illustrate the difference (pun intended)

I drove to charleston (120 mi away)
it took me 3-hours



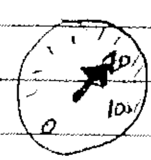
average speed
(rate of change)

$$\frac{\Delta y}{\Delta x} = \frac{120-0}{3-0} =$$

a line that represents the average

the instaneous rate of change is like looking ^1 min @ your speedometer!

☆ differ colors



that's the derivative!

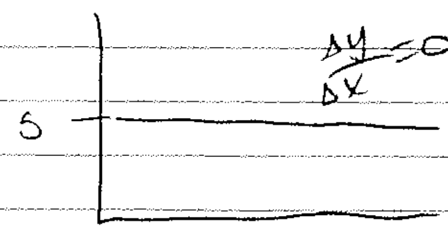
okay let's find the derivative:

* derivative of a constant

$f(x)=k$ then $f'(x)=0$

ex) lets see why!

$f(x)=5$



$\frac{\Delta y}{\Delta x}=0$ always

no matter what x it's a 5

look NO Change!

$f'(x)=0$ means No change!

* can you give me an example of business that might have constant cost?

* constant times a function:

$$\frac{d}{dx}\left[c\cdot f(x)\right] = c\,\frac{d}{dx}[f(x)] = c\cdot f'(x)$$

* Sum or difference:

$$\frac{d}{dx}\left[f(x)+g(x)\right] = f'(x)+g'(x)$$

* power Rule:

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

lines have a constant derivative

i.e. constant speed/change

just like they have
constant average rate of change
they have constant
instantaneous rate of chan

ex)

$$f(x) = 2x+3 = 2x^1 + 3x^0$$

$$f'(x) = 2(1 \cdot x^{1-1}) + 3(0x^{0-1})$$

$$= 2x^0 + 0$$

$$= 2$$

no matter what x
I plug in!

now we can take almost any derivative:

ex) $f(x) = x^{2}$ ← color blue

$f'(x) = 2x^{2-1} = 2x$ ← red

ex) $h(x) = x^3$

$h'(x) = 3x^{3-1} = 3x^2$

ex) $h(x) = 2x^3$

$h'(x) = 2 \frac{d}{dx}[x^3] = 2 \cdot (3x^2) = 6x^2$

ex) $g(x) = 3x^2 + 2x$ $g'(x) = \frac{d}{dx}[3x^2] + \frac{d}{dx}[2x]$

$g'(x) = 6x + 2x^{1-1} = 6x\ 2x^0 = 6x + 2$

ex) $f(x) = 3x^3 + 4x^2 + 5x + 2$ ← derivative is zero!

$f'(x) = 9x^2 + 8x + 5$

ex) $h(x) = \frac{1}{x^2} = x^{-2}$

$h'(x) = -2x^{-2-1} = -2x^{-3}$

ex) $g(x) = x^2 + 3x + 2 \cdot \frac{1}{x}$

$= x^2 + 3x + 2x^{-1}$

$g'(x) = 2x + 3 - 2x^{-2}$

---

ex) $f(x) = \sqrt{x} = x^{1/2}$

$f'(x) = \frac{1}{2} x^{1/2 - 1} = \frac{1}{2} x^{-1/2} = \frac{x^{-1/2}}{2} = \frac{1}{2\sqrt{x}}$

---

ex) $g(x) = x^{4.5}$

$g'(x) = (4.5) x^{4.5-1} = 4.5 x^{3.5}$

---

ex) $h(x) = \frac{1}{\sqrt{x}} = x^{-1/2}$

$h'(x) = -\frac{1}{2} x^{-1/2 - 1} = -\frac{1}{2} x^{-3/2}$

derivative of exponentials & logarithms

$$\frac{d}{dx}[e^x] = e^x$$

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

ex] $f(x) = 3e^x$

$f'(x) = 3e^x$

ex] $h(x) = 2\ln(x)$

$h'(x) = \frac{2}{x}$

yep it's that easy!

* Exponents have

constant relative derivatives

i.e.

defn

The relative (percent) instantaneous change is

$$\frac{f'(x)}{f(x)}$$ looks a lot like

$$\frac{\left(\frac{\Delta f}{\Delta x}\right)}{f(x_1)}$$ → the derivative → the funct

ex)

$f(x) = e^x$

$f'(x) = e^x$

So

$$\frac{f'(x)}{f(x)} = \frac{e^x}{e^x} = 1$$ ← no matter the x

The chain Rule

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

derivative of the outside — derivative of the inside

ex] $f(x) = (\boxed{x^2-3x})^2$ inside the "square" so that's the outer

$$f'(x) = 2(x^2-3x)^{2-1} \cdot (2x-3)$$

$$= 2(x^2-3x)(2x-3)$$

ex] $f(x) = e^{3x^2-2}$ inside the exponent

$$f'(x) = e^{3x^2-2}(6x)$$

ex) $h(x) = \ln(x^3-2x^2+1)$ inside the natural log

$$h'(x) = \frac{1}{x^3-2x^2+1}$$

That's it that's all we do!

powers, exponents, log's

that's All!

that's it! that's it!

product Rule:

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

derivative of the first times the second plus the first times the derivative of the second

ex) $\sqrt{x}\cdot e^{2x} = f(x)$

$$f'(x) = \tfrac{1}{2}x^{-1/2}e^{2x} + \sqrt{x}\left(2e^{2x}\right)$$

ex] $(x^2+3x+2)(x^3+4x^2+x) = y$

$$\frac{dy}{dx} = (2x+3)(x^3+4x^2+x) + (x^2+3x+2)(3x^2+8x+1)$$

Quotient Rule:

$$\frac{\text{lo dhi} - \text{hi dlo}}{\text{lo} \cdot \text{lo}}$$

There Is No quotient Rule!

ex) $f(x) = \dfrac{2x^2+3}{\sqrt{x}}$ ← hi, ← lo

$$f'(x) = \frac{\underbrace{\sqrt{x}}_{\text{lo}}\underbrace{(4x)}_{\text{dhi}} \underbrace{-}_{\text{minus}} \underbrace{(2x^2+3)}_{\text{hi}}\underbrace{\left(\tfrac{1}{2}x^{-1/2}\right)}_{\text{di lo}}}{\underbrace{(\sqrt{x})^2}_{\text{lo lo}}}$$

or

$$f(x) = \frac{2x^2+3}{\sqrt{x}} = (2x^2+3)(x^{-1/2})$$

product rule

$$(4x)(x^{-1/2}) + (2x^2+3)(-\tfrac{1}{2}x^{-3/2})$$

Boom!

group work: (find the derivative)

1] $g(x) = \dfrac{3}{x^2} + 7x^4 + e^{2x} + \ln(3x^2-1)$

2] $S(x) = \left[e^{3x^2+2} - \ln(2x)\right]^3$

3] $R(x) = \sqrt{5x^2 + \dfrac{e^x}{x} + \ln(2x^2+5x)}$

4] $h(x) = \dfrac{2x^2+e^{3x}}{5x^3-7}$

4] $g(x) = \dfrac{e^{3x^2+5x}}{x^2+3x}$

5] $S(x) = \sqrt{3x^2 + \sqrt{x} + \ln(3x^2+2)}$

6] $R(x) = \dfrac{1}{x^7} + x^{3/2} - 2e^{3x^2} - \ln(2x)$

7] $h(x) = \dfrac{\sqrt{x^2+7x+e^{3x}}}{\ln\left[2x^2+e^{4x^2}+7\right]}$

units of "y"

units of "x"

So like mi per hour

~~example: role units~~

~~The cost "C" (in dollars) of building a house that is "A" sq feet is given by the function~~

~~$C = C(A)$~~

~~So $f'(A) = \frac{dC}{dA}$~~

~~So it's cost divided by area~~

~~or~~

~~cost in dollars per square feet!~~

---

ex) So say

the cost to make emoji pillows is modeled by

$$C(q) = 500 + 20q$$

where q is $\boxed{?}$ ← answer this!

$$\frac{d}{dq}[C(q)] = \frac{\text{cost}}{\text{quantity}}$$ so it's cost per pillow made

$$= 20$$

* called Marginal cost *

cost to produce "one more"

* So far this has been constant i.e. a line, but doesn't need to be! *

Say the Revenue function is

$$R(q) = 30q$$

$$\frac{d}{dq}[R(q)] = \frac{\text{Revenue}}{\text{quantity}}$$ Revenue per pillow made

$$= 30$$

* called Marginal Revenue *

now profit is then

$$P(q) = R(q) - C(q)$$

$$= 30q - (500 + 20q)$$

$$= 10q - 500$$

and

$$\frac{d}{dq}[P(q)] = \frac{\text{profit}}{\text{quantity}}$$ profit per pillow made

$$= 10$$

* called Marginal profit *

profit Margin!

* when things are this its really easy to examine the "margins"

So why do companies look
@ Marginal
or "at the margin"

(marginal cost)
the margin tells you per
"The cost of making ^ what you are making
@ this quantity"

or

per what you are making
"the revenue made ^ at this quantity"
or

per what you are making
"the profit made ^ at this quantity"

for example
So ^ if a company wants to if it's
profitable to make another good
they look at the margins

---

ex)

let's say an airline has fixed cost \$400
for each flight q they have to pay
2 pilots at \$200 a flight and 2 flight attendants
at \$100 a flight and say they found that
the maintenance cost is modelled by the func

$$M(q) = 2^q = e^{\ln(2) \cdot q}$$

and say they on average get \$460 a flight

So revenue is easy $R(q) = 460q$

but $C(q) = 400 + 200q + 100q + M(q)$

$$= 400 + 300q + e^{\ln(2) \cdot q}$$

* first ask
if they make a
profit @
10 flights?

if this company is offering 10 flights
* under the assumption that they will
at least get the average price
if they offer an addition flight
should they?

$$R'(q) = 460 \leftarrow \text{constant}$$

$$C'(q) = 300 + \ln(2)\, e^{\ln(2)\cdot q}$$

the cost
at the margin
of 10
flights
we have
now!

So $$C'(10) = 300 + \ln(2)\, e^{\ln(2)\cdot 10}$$

$$= 300 + \ln(2)\cdot 1024$$

$$= 1{,}009.78 > 460$$

i.e.
how
much
we
are paying
per
flight
right
now!

the cost is bigger than the gain!

Shouldn't do it!

see preview page!

intervals of increasing & decreasing

$(2x-4)(x+3)$

$2x^2+6x-4x-12$

$2x^2+2x-12$

* the function is increasing when $f'>0$

and decreasing when $f'<0$

ex) $f(x) = \frac{2}{3}x^3 + x^2 - 12x + 500$

$f'(x) = 2x^2+2x-12$

$0 = 2x^2+2x-12$

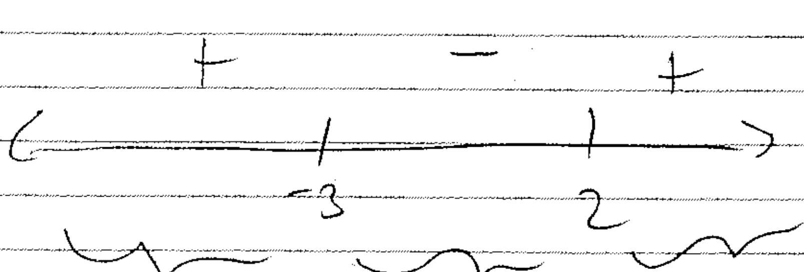
Recall:
$Ax^2+Bx+C=0$
$x = \frac{-B \pm \sqrt{B^2-4AC}}{2A}$

$$x = \frac{-2 \pm \sqrt{4+4\cdot2\cdot12}}{4}$$

$$= \frac{-2 \pm 2\sqrt{25}}{4} = \frac{-2 \pm 10}{4} = \frac{-2+10}{4} = 2$$

or

$$\frac{-2-10}{4} = -3$$



so I know it doesn't change here

so incr $(-\infty, -3)$ and $(2, \infty)$
$\cup$

dec: $(-3, 2)$

local/global
Maxima & minima

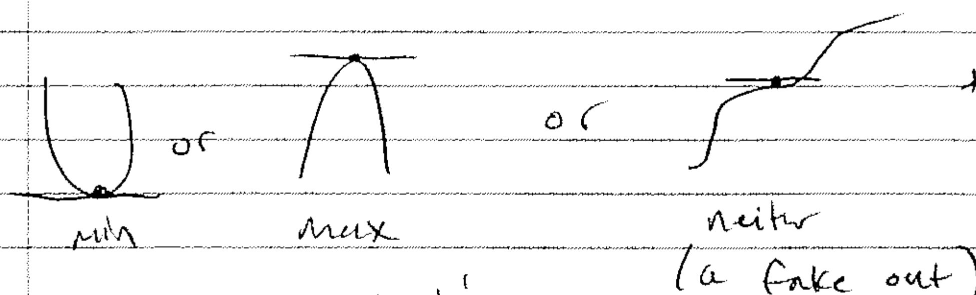
Recall:

$f' > 0$ increase

$f' < 0$ decrease

$f' = 0$ neither

So a neither can look like



* difference between local & global

* stress local!

So we can test for these!

First derivative Test:

Step #1: find the first derivative

~~Step #2: find where step 1 equals zero~~

These are called your (critical points)

Step #3: find out where positive & negative

Step #4: local max & min happen where sign switches i.e. positive – neg or neg – pos

Step #5) plug back into original to find val example: find the local max/min

$$f(x) = 7x^2 + 2x - 5$$

step #1] $f'(x) = 14x + 2$

Step #2] $0 = 14x + 2$

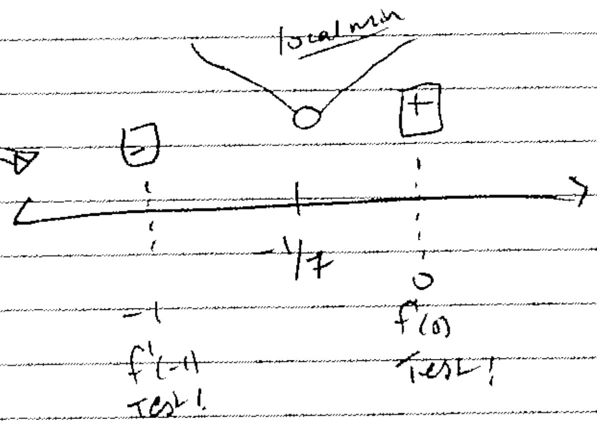
$-2 \qquad -2$

$$\frac{-2}{14} = \frac{14x}{14}$$

$$-\frac{1}{7} = x$$

critical point!

Step #3]



local min

is the global?

so there $(-1/7, \infty)$ dne $(-\infty, -1/7)$

~~Step #4]~~ Step #5] $f(-\frac{1}{7}) = 7(-\frac{1}{7})^2 + 2(-\frac{1}{7}) - 5$

$$= \frac{1}{7} - \frac{2}{7} - 5$$

$$= -\frac{1}{7} - 5 = -\frac{36}{7}$$

local min happens @ $(-\frac{1}{7}, -\frac{36}{7})$

Example: find the local max/min of

$$f(x) = x^3 + 4x^2 - 3x + 7$$

Step #1

$$f'(x) = 3x^2 + 8x - 3$$

Step #2

$$0 = 3x^2 + 8x - 3$$

Recall quadratic eqn $Ax^2 + Bx + C$

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

is the global?

So $x = \dfrac{-8 \pm \sqrt{64 + 36}}{6} = \dfrac{-8 \pm \sqrt{100}}{6} = \dfrac{-8 \pm 10}{6}$

write max or min/as

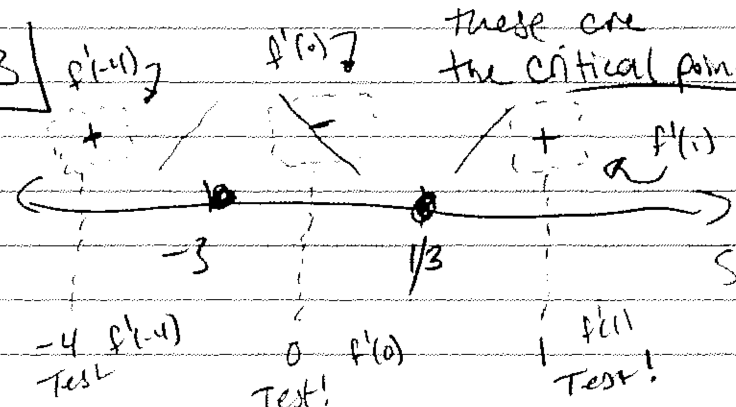
$= \dfrac{-8+10}{6} = \dfrac{2}{6} = \dfrac{1}{3}$ or $= \dfrac{-8-10}{6} = \dfrac{-18}{6} = -3$

these are the critical points!

So local max: $f(-3) =$

local min: $f(1/3) =$

Step #3



pos-neg switch @ -3 the local max happens here

or @ 1/2 neg-pos local min

Skratch: (how I make up question that look pretty)

$$(3x-6)(2x+4) = 6x^2+12x-12x-24$$

ex] find local max/min

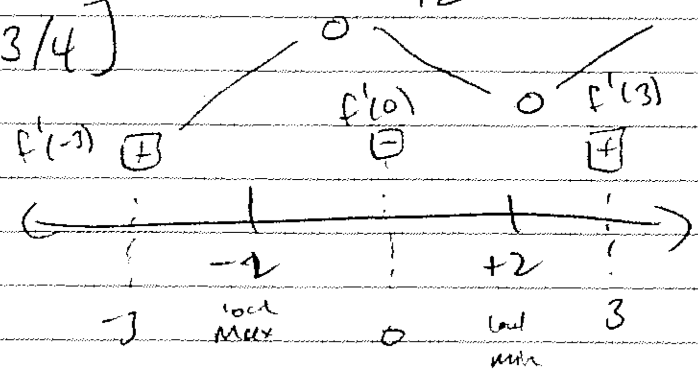
$$f(x) = 2x^3-24x+700$$

Step #1] $f'(x) = 6x^2 = 24$

Step #2] $0 = 6x^2-24$

$$x = \frac{0 \pm \sqrt{4 \cdot 6 \cdot 24}}{12} = \frac{\pm 24}{12} = \pm 2$$

Step #3/4]



☆ is two quad?

☆ write max or min/der

Step #5] $f(-2) = 2(-2)^3-24(-2)+700$

$= -64+48+700 =$ ____

local max @ (-2, ____)

$f(2) = 2(2^3)-24(2)+700$

$64-48+700 =$ ____

local min @ (2, ____)

group work:

{Find incr & decr. global/local Max & min}

$g(x) = 7x^3 - 2x^2 + 5x - 5$

$s(x) = x^2 + 2x - x^3$

$e(c) = 3c^2 - 1$

$h(p) = 3x - e^x$

group work

$\boxed{\text{find incr/decr} \quad \text{global/local Max}}$

$g(r) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x - 700$

$s(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 20x + 2$

$R(c) = 9c^3 - c + 7$

$u(p) = \frac{1}{3}p^3 - p$

ex) find the quantity which maximizes profit

if $R(q) = 5q - 0.003\,q^2$

$C(q) = 300 + 1.1q$

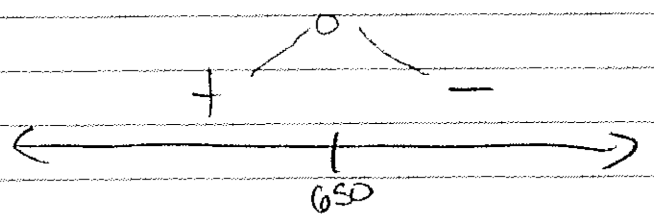
$p(q) = -0.003q^2 + 3.9q - 300$

how could ~~this make~~ sense in a Business? ✡

$p'(q) = -0.006q + 3.9$

$0 = -0.006q + 3.9$

$\frac{-3.9}{0.006} = q$



max profit happens @

$(650, 967.5)$

ex] At a price of \$80 for a half-day trip, a white water rafting company attracts 300 customers. Every \$5 decrease in price attracts an additional 35 customers. [Assuming revenue is a line] find the price which maximizes revenue.

{ • first we need to find the demand function }

{ * why would you know that? * }

$$\frac{D(80) - D(75)}{80 - 75} = \frac{300 - 335}{5} = -7 \quad \text{* Slope!}$$

$$q - 335 = -7(p - 75)$$

$$q = -7p + 860$$

{ • Now Revenue }

function of quantity

$$R(q) = p \cdot q$$

function of price

$$R(p) = R(D(p)) = p \cdot D(p) = p(-7p + 860) = -7p^2 + 860p$$

$$R'(p) = -14p + 860, \quad 0 = -14p + 860$$

$$p = \frac{860}{14} = 61.43$$

$+$ $-$

61.43

377365

52,879.8

max happens @

$p = 61.43$

and $R(61.43) \approx \$49{,}056.^{15}$

The second derivative:

(Acceleration)

The change of the change!

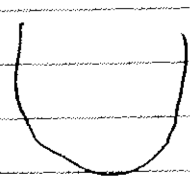
Notation: $f''(x)$ or $\frac{d^2y}{dx^2}$

* It's the $\boxed{\text{derivative}}$ of the derivative *

so we have tools to
say when the derivative
is
increasing & decreasing
and
global/local Max & min

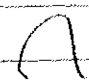
when the first derivative is increasing
the original function is
what's called

Concave up:



the bowl is up

when the dr. is decr the original fnct
is what is called
concave down



the bowl is down

ex) find where f(x) is concave up & concave down
~~(i.e. where the derivative is incr & decr)~~

$$f(x) = 2x^3 - x^2 + 2x - 5$$

Step #1) ~~find second derivative~~

$$f''(x) = ?$$

$$f'(x) = 6x^2 - 2x + 2$$

so

$$f''(x) = 12x - 2$$

Step #2) where is the second der 0?

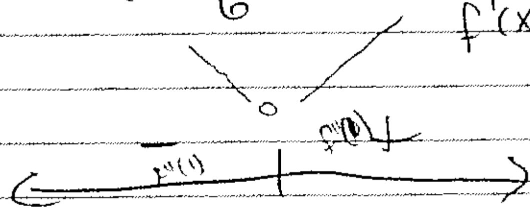
$$0 = 12x - 2$$

$$x = \frac{1}{6}$$

* color

$f'(x)$

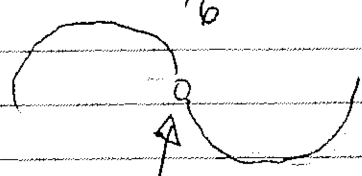
Step #3)



$1/6$

* diff color

$f(x)$



called inflection points & where it changes

CU: $(1/6, \infty)$

CD: $(-\infty, 1/6)$

ex] find where the function is concave up and concave down (i.e. where the derivative is incr & decr)

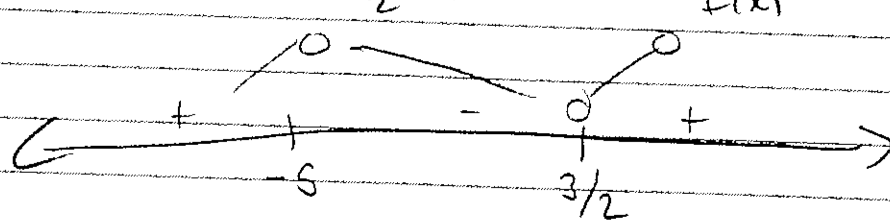
$$f(x) = \frac{1}{6}x^4 + \frac{1}{2}x^3 - \frac{15}{2}x^2 + 3x + 5$$

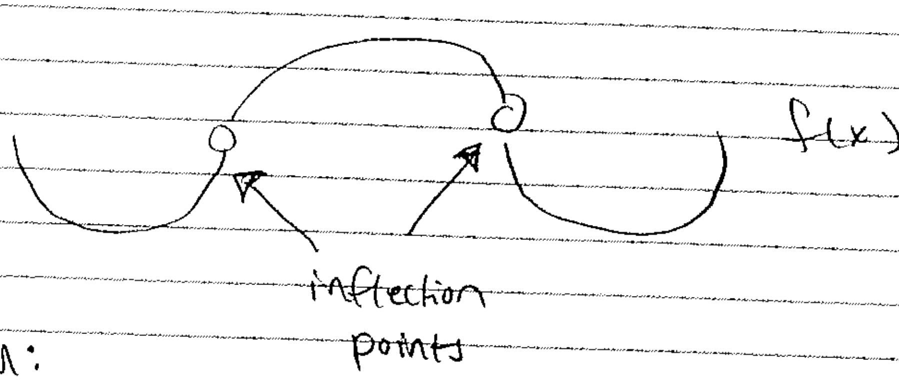
$$f'(x) = \frac{2}{3}x^3 + x^2 - 15x + 3$$

$$f''(x) = 2x^2 + 2x - 15$$

$$0 = 2x^2 + 2x - 15$$

$$x = \frac{3}{2} \text{ or } -5$$





CU:

$(-\infty, -5) \cup (3/2, \infty)$

CD:

$(-5, 3/2)$

{# recall it remains profitable to make "1-more" whenever the $\underbrace{\text{marginal profit}}_{\text{profit margin}}$ is positive}

{assuming they can sell that quantity}

ex) if the profit function for some company is modeled by

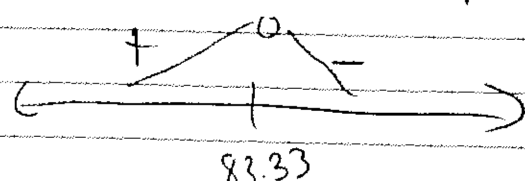
$$p(q) = -3q^2 + 500q - 300$$

What quantity should this company produce so that at any less quantity it would be profitable to make more and any larger quantity would not be profitable.

{what am I asking you??}

$$p'(q) = -6q + 500$$

$$0 = -6q + 500, \quad q = \frac{500}{6} \approx 83.33$$



* Is this the {Break even point?} *
why or why not?

So making 83.33 would do that since any less and the profit margin would be positive

and any more and the profit margin would be negative!