[1]

ADJOINT FUNCTORS AND BALANCING Tor AND Ext

KELLER VANDEBOGERT

1. Adjoint Functors

Definition 1.1 (Adjoint Functors)**.** Let

$$L : \mathcal{A} \to \mathcal{B}, \quad R : \mathcal{B} \to \mathcal{A}$$

The pair of functors L and R are called adjoint if there is a natural isomorphism

$$\Phi : \mathrm{Hom}_{\mathcal{B}}(L(-), -) \to \mathrm{Hom}_{\mathcal{A}}(-, R(-))$$

That is, given $f : \mathcal{A} \to \mathcal{A}'$ and $g : \mathcal{B} \to \mathcal{B}'$, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{B}}(L(A'), B) \arrow[r, "Lf^*"] \arrow[d, "\Phi"'] & \mathrm{Hom}_{\mathcal{B}}(L(A), B) \arrow[r, "g_*"] \arrow[d, "\Phi"'] & \mathrm{Hom}_{\mathcal{B}}(L(A), B') \arrow[d, "\Phi"'] \\
\mathrm{Hom}_{\mathcal{A}}(A', R(B)) \arrow[r, "f^*"] & \mathrm{Hom}_{\mathcal{A}}(A, R(B)) \arrow[r, "Rg_*"] & \mathrm{Hom}_{\mathcal{A}}(A, R(B'))
\end{tikzcd}$$

Theorem 1.2. *Let*

$$L : \mathcal{A} \to \mathcal{B}, \quad R : \mathcal{B} \to \mathcal{A}$$

be an adjoint pair of additive functors. Then L is right exact and R is left exact.

Proof. Choose an exact sequence

$$0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$$

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of objects in $\mathcal{B}$. The functor $\mathrm{Hom}_{\mathcal{B}}(L(A), -)$ is covariant and left exact, so that

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{B}}(L(A), B') \longrightarrow \mathrm{Hom}_{\mathcal{B}}(L(A), B) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(L(A), B'')$$

remains exact; using the natural isomorphism for our adoint pair,

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A, R(B')) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A, R(B)) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A, R(B''))$$

is exact. As $A \in \mathcal{A}$ is arbitrary, the Yoneda Lemma gives that

$$0 \longrightarrow R(B') \longrightarrow R(B) \longrightarrow R(B'')$$

is also exact, and we see that R is left exact. Applying the above argument to L^{op} shows that L^{op} is also left exact, so that L is right exact.

$\square$

Proposition 1.3 (Hom-Tensor Adjointness/Adjunction)**.** *Let A be a right R module, B an (R, S) bimodule, and C a right S module. Then*

$$\mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)) \cong \mathrm{Hom}_S(A \otimes_R B, C)$$

naturally; that is, $- \otimes_R B$ and $\mathrm{Hom}_S(B, -)$ are an adjoint pair.

Proof. Define $\tau : \mathrm{Hom}_S(A \otimes_R B, C) \to \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$ as the map such that for $f \in \mathrm{Hom}_S(A \otimes_R B, C)$,

$$(\tau f)(a)(b) = f(a \otimes b)$$

Now the association $(a, b) \mapsto g(a)(b)$ for $g \in \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$ is bilinear, hence induces a map $f : A \otimes_R B \to C$; we define $\tau^{-1}g$ to be this induced map, so that τ is an isomorphism (it is easy to see that this is our actual inverse).

It remains to show the naturality conditions. Given A, $A' \in \mathcal{A}$, B, $B' \in \mathcal{B}$ and $f : A \to A'$, $g : C \to C'$, let $\phi \in \mathrm{Hom}_{\mathcal{B}}(A' \otimes_R B, C)$. Then, for $a \in A$, $b \in B$,

$$\begin{aligned}\tau(f \otimes B)^*(\phi)(a)(b) &= (f \otimes B)^*\phi(a \otimes b) \\ &= \phi(f \otimes B)(a \otimes b) \\ &= \phi(f(a) \otimes b)\end{aligned}$$

Similarly,

$$\begin{aligned}f^*(\tau\phi)(a)(b) &= (\tau\phi)(f(a))(b) \\ &= \phi(f(a) \otimes b)\end{aligned}$$

So the first square in the diagram of Definition 1.1 commutes. Now let us consider the second square. Given $\psi \in \mathrm{Hom}_{\mathcal{B}}(A \otimes_R B, C)$, $a \in A$, and $b' \in B'$,

$$\tau(g_*\psi)(a)(b') = g(\psi(a \otimes b'))$$

and

$$\begin{aligned}(g_*)_*(\tau\psi)(a)(b') &= g_*(\phi(a \otimes b') \\ &= g(\phi(a \otimes b'))\end{aligned}$$

So the second square also commutes, and we get naturality. □

The above gives us a small corollary, although this can be proved by more elementary means.

Corollary 1.4. $\mathrm{Hom}(A, -)$ *is left exact and* $- \otimes B$ *is right exact.*

Definition 1.5. If B is a left R module and A is a right R module, define $T(A) = A \otimes_R B$. Then,

$$\mathrm{Tor}_i^R(A, B) := (L_n T)(A)$$

Similarly, if we define $E(B) = \mathrm{Hom}(A, B)$, then

$$\mathrm{Ext}^i_R(A, B) := (R^n E)(B)$$

Any diagram of objects in a category $\mathcal{C}$ can be viewed as a functor $F : \mathcal{I} \to \mathcal{C}$, where $\mathcal{I}$ can be interpreted as the indexing or diagram category.

We have a functor $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$, where $\Delta(Q) = \Delta_Q$, and Δ_Q is the trivial functor sending every object in $\mathcal{I}$ to Q and every morphism to id_Q.

Definition 1.6 (Limits)**.** The limit of a functor $F : \mathcal{I} \to \mathcal{C}$ is an object $P \in \mathcal{C}$ and a natural transformation $\Delta_P \to F$ with the following universal property: given any other natural transformation $\Delta_Q \to F$, there exists a unique map $f : Q \to P$ making the following diagram commute

$$\begin{tikzcd} & \Delta_P \arrow[d] \\ \Delta_Q \arrow[ur, "\Delta_f"] \arrow[r] & F \end{tikzcd}$$

In typical fashion, we may dualize all of the above to find the definition of colimits.

Definition 1.7 (Colimits)**.** A colimit of a diagram $F : \mathcal{I} \to \mathcal{C}$ is an object $P \in \mathcal{C}$ and a morphism of diagrams $F \to \Delta_P$ that is initial among all morphisms $F \to \Delta_Q$ for $Q \in \mathcal{C}$. That is, given $Q \in \mathcal{C}$ also satisfying the above, there exists $f : P \to Q$ such that the following diagram commutes

$$\begin{tikzcd} & \Delta_P \arrow[dl, "\Delta_f"'] \\ \Delta_Q & F \arrow[l] \arrow[u] \end{tikzcd}$$

Proposition 1.8. *Let $\mathcal{A}$ be an abelian category; the following are equivalent:*

1. *The direct sum $\bigoplus A_i$ exists for every set of object $\{A_i\}$, $A_i \in \mathcal{A}$.*
2. *$\mathcal{A}$ is cocomplete; that is $\operatorname{colim}_{i\in I} A_i$ exists in $\mathcal{A}$ for each functor $A : I \to \mathcal{A}$ whose indexing category I has only a set of elements.*

Proof. Note that (1) follows immediately from (2) since the direct sum is itself a colimit.

If (1) holds, then we consider the cokernel of

$$\begin{gathered}\bigoplus_{\phi: i\to j} A_i \to \bigoplus_{i\in I} A_i \\ a_i(\phi) \mapsto \phi(a_i) - a_i\end{gathered}$$

is precisely $\operatorname{colim}_{i\in I} A_i$. □

Theorem 1.9 (Adjoints and Limits Theorem)**.** *Let $L : \mathcal{A} \to \mathcal{B}$ be left adjoint to a functor $R : \mathcal{B} \to \mathcal{A}$, where $\mathcal{A}$ and $\mathcal{B}$ are arbitrary categories. Then,*

1. *L preserves all colimits. That is, if $A : I \to \mathcal{A}$ has a colimit, then so does $L(A) : I \to \mathcal{B}$, and*
$$L(\operatorname{colim}_{i\in I} A_i) = \operatorname{colim}_{i\in I} L(A_i)$$
2. *R preserves all limits. That is, if $B : I \to \mathcal{B}$ has a limit, then so does $R(B) : I \to \mathcal{A}$, and*
$$R(\lim_{i\in I} B_i) = \lim_{i\in I} R(B_i)$$

Proof. Note that we only need prove the theorem for the left adjoint L; applying the argument to R^{op} yields (2).

We start with the following observation: if $\{A_i, \phi^i_j\}$ is a directed system in $\mathcal{A}$ over $\mathcal{I}$, then $\{LA_i, L\phi^i_j\}$ is also a directed system. By the universal property of colimits, we have the existence of morphisms making the following diagram commute

$$\begin{tikzcd}
L(\operatorname{colim}_{i\in I} A_i) & & \operatorname{colim}_{i\in I}(LA_i) \\
& LA_i \arrow[ul, "L\alpha_i"'] \arrow[ur, "f_i"] \arrow[d, "L\phi^i_j"] & \\
& LA_j \arrow[uul, "L\alpha_j"] \arrow[uur, "f_j"'] &
\end{tikzcd}$$

Now, we want to show that there exists a natural map $L(\operatorname{colim}_{i\in I} A_i) \to \operatorname{colim}_{i\in I}(LA_i)$ making the above commute. We are already guaranteed a map backwards by the universal property of colimits. To this end, consider applying R to the above diagram. The key here is to note that there is a natural transformation $\eta : 1_{\mathcal{A}} \to RL$ (the unit of the adjoint pair); this will translate the diagram back to a diagram in $\mathcal{A}$, and the universal property of colimits will guarantee our map. Applying R and using naturality of the unit, we have the induced commutative diagram

$$\begin{tikzcd}
\operatorname{colim}_{i\in I} A_i & & R(\operatorname{colim}_{i\in I}(LA_i)) \\
& A_i \arrow[ul, "\alpha_i"'] \arrow[ur, "Rf_i"] \arrow[d, "\phi^i_j"] & \\
& A_j \arrow[uul, "\alpha_j"] \arrow[uur, "Rf_j"'] &
\end{tikzcd} \tag{1.1}$$

whence there exists a map $\beta : \operatorname{colim}_{i\in I} A_i \to R(\operatorname{colim}_{i\in I}(LA_i))$. Since L and R are an adjoint pair, we have a natural isomorphism

$$\tau : \operatorname{Hom}_{\mathcal{B}}(L(\operatorname{colim}_{i\in I} A_i), \operatorname{colim}_{i\in I}(LA_i)) \to \operatorname{Hom}_{\mathcal{A}}(\operatorname{colim}_{i\in I} A_i, R(\operatorname{colim}_{i\in I}(LA_i))$$

Since $\beta \in \operatorname{Hom}_{\mathcal{A}}(\operatorname{colim}_{i\in I} A_i, R(\operatorname{colim}_{i\in I}(LA_i))$, we may define $\tau^{-1}\beta :=$ $\gamma \in \operatorname{Hom}_{\mathcal{B}}(L(\operatorname{colim}_{i\in I} A_i), \operatorname{colim}_{i\in I}(LA_i))$.

It remains to show that γ makes the following diagram commute:

$$\begin{tikzcd}
L(\operatorname{colim}_{i\in I} A_i) \arrow[rr, "\gamma"] & & \operatorname{colim}_{i\in I}(LA_i) \\
& LA_i \arrow[ul, "L\alpha_i"'] \arrow[ur, "f_i"] \arrow[d, "L\phi^i_j"] & \\
& LA_j \arrow[uul, "L\alpha_j"] \arrow[uur, "f_j"'] &
\end{tikzcd}$$

That is, $\gamma(L\alpha_i) = f_i$ for every $i \in I$. Rewriting, we see

$$\begin{aligned}
\gamma(L\alpha_i) &= \tau^{-1}\beta)(L\alpha_i) \\
&= (L\alpha_i)^*(\tau^{-1}\beta) \qquad (1.2)\\
&= \tau^{-1}(\alpha_i)^*\beta
\end{aligned}$$

Where the last equality follows by commutativity of the first square of the diagram given in Definition 1.1. We consider the term $\tau^{-1}(\alpha_i)^*\beta = \tau^{-1}(\beta\alpha_i)$. By commutativity of the diagram (1.1), $\beta\alpha_i = Rf_i$, ; we may now use commutativity of the second square in Definition 1.1 to see that $\tau(f_i)^* = (Rf_i)^*\tau$; evaluating at the identity yields $\tau(f_i)_*(1) = (Rf_i)_*$, whence

$$\begin{aligned}
\gamma(L\alpha_i) &= \tau^{-1}(\beta\alpha_i) \\
&= \tau^{-1}(Rf_i)_* \\
&= \tau^{-1}(\tau(f_i)_*(1)) \\
&= f_i
\end{aligned}$$

And we get commutativity. Uniqueness of γ also follows immediately by the universal property of our colimits. Therefore we have constructed a natural map $\gamma : \operatorname{Hom}_{\mathcal{B}}(L(\operatorname{colim}_{i\in I} A_i), \operatorname{colim}_{i\in I}(LA_i))$; the inverse map is the induced map from the universal property of colimits, and we have a natural isomorphism

$$L(\operatorname{colim}_{i\in I} A_i) \cong \operatorname{colim}_{i\in I}(LA_i)$$

□

The above immediately yields the following corollaries:

Corollary 1.10. *If a cocomplete abelian category $\mathcal{A}$ has enough projectives and $F : \mathcal{A} \to \mathcal{B}$ is a left adjoint functor, then for every set $\{A_i\}$ of objects in $\mathcal{A}$,*

$$L_*F\Big(\bigoplus_{i\in I} A_i\Big) = \bigoplus L_*F(A_i)$$

Proof. Observe that if we have projective resolutions $P_i \to A_i$, then $\bigoplus_{i\in I}$ is a projective resolution for $\bigoplus_{i\in I} A_i$. Then

$$L_*F(\bigoplus_{i\in I} A_i) = H_*(F(\bigoplus_{i\in I} A_i))$$

By the previous theorem, F preserves colimits; in particular, $F(\bigoplus_{i\in I} A_i) = \bigoplus_{i\in I} F(A_i)$, and

$$\begin{aligned} L_*\Big(\bigoplus_{i\in I} F(A_i)\Big) &= H_* \bigoplus_{i\in I} F(A_i) \\ &= \bigoplus_{i\in I} H_*F(A_i) \\ &= \bigoplus_{i\in I} L_*F(A_i) \end{aligned}$$

□

Corollary 1.11.

$$\mathrm{Tor}_i^R\big(A, \bigoplus_{i\in I} B_i\big) = \bigoplus_{i\in I} \mathrm{Tor}_i^R(A, B_i)$$

We shall now present some conditions for particular functors to have a particular representation; first we will need the following

Definition 1.12. If M is a right R-module and $m \in M$, then $\phi_m : R \to M$ is defined by the map $r \mapsto mr$.

The above map takes advantage of the natural isomorphism $M \cong \mathrm{Hom}_R(R, M)$; this point of view will be useful in the following theorems. In what follows, one can imagine these theorems of being the homological analogues of the classical representation theorems of analysis such as the Riesz representation theorem and the Radon-Nikodym theorem. Instead of considering integral representations of measures and linear operators, we want representations of an arbitrary functor in terms of the standard Hom and tensor functors.

Theorem 1.13. *If $F : \boldsymbol{Mod}_R \to \boldsymbol{Ab}$ is a right exact additive functor that preserves direct sums, then F is naturally isomorphic to $- \otimes_R B$, where $B = F(R)$, where $F(R)$ is given the natural structure of a right R-module.*

Proof. Let us first give $F(R)$ an R-module structure. As ϕ_m as above is an element of mod $_R(R, M)$, we see that $F\phi_m \in \mathrm{Hom}_{\mathbb{Z}}(F(R), F(M))$. For $M = R$, we have a natural action on R where, given r, $x \in R$,

$$rx := (F\phi_r)(x)$$

Let us check associativity first. Let r, s, $x \in R$:

$$\begin{aligned}(rs)x &= (F\phi_{rs})(x) \\ &= (F\phi_r\phi_s)(x) \\ &= (F\phi_r F\phi_s)(x) \\ &= r(sx)\end{aligned}$$

The rest of the properties are trivial as the ϕ_r are homomorphisms of abelian groups. Now define $F(R) := B$.

Consider the map

$$\tau_M : M \times B \to F(M)$$

defined by sending $(m, x) \mapsto (F\phi_m)(x)$. This is R-bilinear (additivity is clear since F is additive) and

$$\begin{aligned}
\tau_M(mr, x) &= (F\phi_{mr})(x) \\
&= (F\phi_m F\phi_r)(x) \\
&= (F\phi_m)ux = \tau_M(m, rx)
\end{aligned}$$

and we have an induced map

$$\psi_M : M \otimes_R B \to F(M)$$

It remains to show that $\psi : - \otimes_R B \to F$ is a natural transformation so that

$$\begin{tikzcd}
M \otimes_R B \arrow[r, "\psi_M"] \arrow[d, "f\otimes B"] & F(M) \arrow[d, "Ff"] \\
N \otimes_R B \arrow[r, "\psi_N"] & F(N)
\end{tikzcd}$$

whenever $f \in \mathrm{Hom}_R(M, N)$. Following the diagram along the top, we see

$$\begin{aligned}
m \otimes x &\mapsto (F\phi_m)(x) \\
&\mapsto (Ff)(F\phi_m)(x) \\
&= F(f\phi_m)(x)
\end{aligned}$$

and, aong the bottom:

$$\begin{aligned}
m \otimes x &\mapsto f(m) \otimes x \\
&\mapsto (F\phi_{f(m)})(x)
\end{aligned}$$

From our definition of ϕ, given any $r \in R$ we see that $\phi_{f(m)}(r) = f(m)r = f\phi_m(r)$, so that $\phi_{f(m)} = f\phi_m$. Comparing the above, we then see that the diagram does commute, yielding naturality of ψ.

Now, as $B = F(R)$ is an R-module, we deduce that ψ_R is an isomorphism. Since F preserves direct sums, ψ_A is an isomorphism for every free right R-module A. Given any R-module M, there exists an exact sequence

$$F_0 \longrightarrow F_1 \longrightarrow M \longrightarrow 0$$

where F_0, F_1 are free (not necessarily finitely generated). By naturality of ψ, we have the following induced commutative diagram with exact rows:

$$\begin{tikzcd}
F_0 \otimes_R B \arrow[r] \arrow[d, "\psi_{F_0}"] & F_1 \otimes_R B \arrow[r] \arrow[d, "\psi_{F_1}"] & M \otimes_R B \arrow[r] \arrow[d, "\psi_M"] & 0 \\
F(F_0) \arrow[r] & F(F_1) \arrow[r] & F(M) \arrow[r] & 0
\end{tikzcd}$$

Since ψ_{F_0} and ψ_{F_1} are isomorphisms, the Five Lemma gives that ψ_M is an isomorphism as well. Thus, ψ is a natural isomorphism. $\square$

Theorem 1.14. *If $H : R - \boldsymbol{Mod} \to \boldsymbol{Ab}$ is a contravariant left exact additive functor that converts direct sums to direct products, then H is naturally isomorphic to $\operatorname{Hom}_R(-, B)$, where $H(R) := B$ is given the natural structure of an R-module.*

Proof. We again start by giving $H(R) =: B$ the structure of a right R-module. We define the action of R in an identical fashion as for the previous proof, but as a right action:

$$xr := (H\phi_r)(x)$$

Then, we see

$$\begin{aligned}
x(rs) &= (H\phi_{rs})(x) \\
&= (H(\phi_r\phi_s))(x) \\
&= (H\phi_s H\phi_r)(x) \quad \text{(contravariance)} \\
&= (H\phi_s)(xr) \\
&= (xr)s
\end{aligned}$$

So this is well defined as a right R-module. Consider now the map $\psi : H \to \mathrm{Hom}_R(-, B)$ where $\psi_M(x) \in \mathrm{Hom}_R(M, B)$ is such that

$$\psi_M(x)(m) := (H\phi_m)(x)$$

Let $f \in \mathrm{Hom}_R(M, N)$; to prove naturality, we must show commutativity of the following diagram:

$$\begin{tikzcd} H(N) \arrow[r, "\psi_N"] \arrow[d, "Hf"] & \mathrm{Hom}_R(N, B) \arrow[d, "f^*"] \\ H(M) \arrow[r, "\psi_M"] & \mathrm{Hom}_R(M, B) \end{tikzcd}$$

That is, we want to show $f^*\psi_N = \psi_M(Hf)$. We have, for $x \in H(N)$, $m \in M$:

$$\begin{aligned} f^*\psi_N(x)(m) &= \psi_N(x)(f(m)) \\ &= (H\phi_{f(m)})(x) \\ &= (H(f\phi_m))(x) \\ &= (H\phi_m Hf)(x) \\ &= H\phi_m(Hf(x)) \\ &= \psi_M(Hf(x))(m) \end{aligned}$$

whence naturality follows. Now, in order to prove that this is an isomorphism, we proceed similarly as in the previous proof. Note that $\psi_R : H(R) \to \mathrm{Hom}_R(R, B)$ is an isomorphism as $\mathrm{Hom}_R(R, B) = \mathrm{Hom}_R(R, H(R)) \cong H(R)$ by the natural map $f \mapsto f(1)$. Hence we deduce that ψ is an isomorphism on free modules, so that given an R-module M, we have the exact sequence

$$F_0 \longrightarrow F_1 \longrightarrow M \longrightarrow 0$$

with F_0, F_1 free; we get the induced commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_R(M,B) & \longrightarrow & \mathrm{Hom}_R(F_1,B) & \longrightarrow & \mathrm{Hom}_R(F_0,B) \\
& \big\downarrow \psi_M & & \big\downarrow \psi_{F_1} & & \big\downarrow \psi_{F_0} \\
0 \longrightarrow & H(M) & \longrightarrow & H(F_1) & \longrightarrow & H(F_0)
\end{array}$$

Employing the Five Lemma, we see that ψ_M must be an isomorphism.

□

We've managed to characterize two of our favorite functors, $- \otimes_R B$ and $\mathrm{Hom}_R(-, B)$. One may ask about the case for $\mathrm{Hom}_R(B, -)$; at first sight, it seems that merely dualizing some of the above argument will do the trick. However, travelling along this path quickly leads to difficulties. Indeed, characterizing the covariant Hom functor will need some extra machinery.

Definition 1.15. A left R-module C is called a cogenerator of $R-\mathbf{Mod}$ if, for every R-module M and every nonzero $m \in M$, there exists $g \in \mathrm{Hom}_R(M, C)$ with $g(m) \neq 0$.

Lemma 1.16. *There exists an injective cogenerator of $R - \boldsymbol{Mod}$.*

A proof of the above is not too difficult. The key, of course, is how to define our cogenerator. To do this, consider

$$\bigoplus_I R/I$$

where the direct sum runs over all ideals of R. We can embed the above module into its injective hull, and call this C. Then, C is the desired injective cogenerator. To see this, let M be any left R-module. Given $m \in M$, note that $Rm \cong R/\operatorname{Ann}(m)$, so we have an inclusion

$$Rm \hookrightarrow C$$

which, by injectivity, induces a map $f \in \operatorname{Hom}_R(M, C)$. Since $m \neq 0$, the image of m in C is nonzero. Since f is an extension, $f(m) \neq 0$ as well, as desired.

Lemma 1.17. *Let*

$$\begin{tikzcd} D \arrow[r, "\alpha"] \arrow[d, "\beta"] & C \arrow[d, "g"] \\ B \arrow[r, "f"] & A \end{tikzcd}$$

be a pullback diagram in $\boldsymbol{Ab}$*. If there is* $c \in C$ *and* $b \in B$ *with* $g(c) = f(b)$*, then there exists* $d \in D$ *such that* $c = \alpha(d)$ *and* $b = \beta(d)$*.*

Proof. Define

$$p : \mathbb{Z} \to C, \quad q : \mathbb{Z} \to B$$

by $p(n) = nc$ and $q(n) = nb$. Then, this has the property that $f(q(n)) = nf(b)$, and $g(p(n)) = ng(c)$, and by assumption $f(b) = g(c)$. By the universal property of pullbacks, there exists a map $\theta : \mathbb{Z} \to D$ with $p = \alpha\theta$, $q = \beta\theta$. Then, define $d := \theta(1)$; we see

$$\alpha(\theta(1)) = p(1) = c, \quad \beta(\theta(1)) = q(1) = b$$

so that $\theta(1)$ is the desired d. □

Lemma 1.18. *Let* $\mathcal{M}$ *be a family of submodules. We may partially order* $\mathcal{M}$ *by reverse inclusion, where our transition maps are the natural inclusions.*

If $\mathcal{M}$ *is closed under finite intersections (that is,* $M, N \in \mathcal{M} \implies M \cap N \in \mathcal{M}$*), then*

$$\bigcap_{M \in \mathcal{M}} M = \varprojlim_{M \in \mathcal{M}} M$$

Proof. Define the inclusion

$$p_M : \bigcap_{M \in \mathcal{M}} M \hookrightarrow M$$

Suppose D, M, $N \in \mathcal{M}$, and let $D \subset M$. Then, when $i_M^D : D \hookrightarrow M$ denotes our inclusion, it is obvious that $p_D = i_M^D p_M$. As $\mathcal{M}$ is closed under finite intersections, we deduce that $D = M \cap N$, and in particular $p_M(x) = p_{M\cap N}(x) = p_N(x)$.

This gives that the map $\phi : \varprojlim_{M \in \mathcal{M}} M \to \bigcap_{M \in \mathcal{M}} M$ with $\phi(x) = p_M(x)$ is well defined and does not depend on the choice of module M. The universal property of limits gives us an inverse map, so we deduce that ϕ is an isomorphism as asserted. □

Definition 1.19. For a module M and a set X, M^X will denote the set of all functions (not necessarily homomorphisms!) from X to M. With this notation, $p_x : M^X \to M$ is the map such that $p_x(f) = f(x)$.

The notation M^X comes from the point of view as viewing a function $f : X \to M$ as an element of the direct product $M^{\prod X}$, where we consider the direct product as being indexed by X. With this view, the map p_x as above is merely the projection onto the "xth" coordinate, and, we also have a natural module structure induced by the module structure given by the direct product.

We are now in a position to handle the covariant Hom case.

Theorem 1.20. *If $G : R - \textbf{\textit{Mod}} \to \textbf{\textit{Ab}}$ is a covariant additive functor that preserves inverse limits, then G is naturally isomorphic to* $\mathrm{Hom}_R(B, -)$ *for some left R-module.*

Proof. Choose an injective cogenerator C as above, and consider the set $\Pi := C^{G(C)}$. G preserves inverse limits, so in particular, it preserves products; whence $G\Pi$ is a direct product with induced projection maps Gp_x. By the universal property of products, there exists a natural map $\theta : G(C)^{G(C)}$ making the following diagram commute:

$$\begin{tikzcd}
G\Pi \arrow[dr, "Gp_x"'] & & G(C)^{G(C)} \arrow[ll, "\theta"'] \arrow[dl, "\pi_x"] \\
& G(C) &
\end{tikzcd}$$

Set $e := 1_{G(C)} \in G(C)^{G(C)}$, and define $\tau : \mathrm{Hom}_R(\Pi, C) \to G(C)$ by

$$\tau(f) = (Gf)(\theta e)$$

We see that τ is surjective, since given $x \in G(C)$,

$$\begin{aligned}
\tau(p_x) &= (Gp_x)(\theta e) \\
&= \pi_x(e) = x
\end{aligned}$$

Now, if S is a submodule of Π, let $i_S : S \hookrightarrow \Pi$ be the natural inclusion. Define

$$\mathcal{S} := \{\text{Submodules } S \subset \Pi \mid \theta(e) \subset \mathrm{Im}\big(G(i_S)\big)\}$$

and set

$$B := \bigcap_{S \in \mathcal{S}} S$$

Under the inclusion maps λ and μ, $S \cap T$ is the pullback in the following diagram

$$\begin{tikzcd}
S \cap T \arrow[r, "\lambda"] \arrow[d, "\mu"] & S \arrow[d, "i_S"] \\
T \arrow[r, "i_T"] & \Pi
\end{tikzcd}$$

And, since G preserves colimits, it preserves pullbacks, so that $G(S \cap T)$ is the pullback in

$$\begin{tikzcd}
G(S \cap T) \arrow[r, "G\lambda"] \arrow[d, "G\mu"] & G(S) \arrow[d, "Gi_S"] \\
G(T) \arrow[r, "Gi_T"] & G(\Pi)
\end{tikzcd}$$

By our definition of the set $\mathcal{S}$, for $u \in G(S)$ and $v \in G(T)$, $(Gi_S)(v) = \theta e$ and $(Gi_T)(v) = \theta e$. Employing Lemma 1.17, we may find $d \in G(S \cap T)$ with $(Gi_S)(G\lambda)(d) = \theta e$.

By covariance, $(Gi_S)(G\lambda)(d) = G(i_S\lambda)(d) = (Gi_{S\cap T})(d)$. But this shows that $\theta e \in \operatorname{Im}\big(G(i_{S\cap T})\big)$, so that $\mathcal{S}$ is closed under finite intersections. By Lemma 1.18,

$$B = \varprojlim_{S\in\mathcal{S}} S$$

In which case $B \in \mathcal{S}$. Let $j : B \hookrightarrow \Pi$ be our inclusion.

Claim: $\operatorname{Ker}\tau = \operatorname{Ker} j^*$. Let $f \in \operatorname{Hom}_R(\Pi, C)$. Suppose first that $f \in \operatorname{Ker}\tau$, so that $(Gf)(\theta e) = 0$. In particular, this means that $\theta(e) \in \operatorname{Ker} Gf$; consider the exact sequence

$$0 \longrightarrow \operatorname{Ker} f \xrightarrow{\iota} \Pi \xrightarrow{f} C$$

Since G is left exact, we have the induced exact sequence

$$0 \longrightarrow G(\operatorname{Ker} f) \xrightarrow{G\iota} \Pi \xrightarrow{Gf} G(C)$$

From which we see that $\operatorname{Ker} Gf = \operatorname{Im} G\iota$, and by definition of $\mathcal{S}$, $\operatorname{Ker} f \in \mathcal{S}$. But this means that $\operatorname{Ker} f \supset B$, in which case $fj = j^*(f) = 0$, whence $f \in \operatorname{Ker} j^*$.

For the reverse inclusion, suppose now that $f \in \operatorname{Ker} j^*$. If $j^*(f) = 0$, in particular we see that B must be contained in the kernel of f. Using the above exact sequences, this implies

$$\operatorname{Ker} Gj \subset \operatorname{Ker} G\iota = \operatorname{Ker} Gf$$

Since $\theta e \in \operatorname{Ker} Gf$, $(Gf)(\theta e) = 0$, and by definition of τ, this means $\tau(f) = 0 \implies f \in \operatorname{Ker}\tau$. This gives that $\operatorname{Ker}\tau = \operatorname{Ker} j^*$.

We have the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_R(\Pi/B, C) \arrow[r] \arrow[d, equal] & \operatorname{Hom}_R(\Pi, C) \arrow[r, "j^*"] \arrow[d, equal] & \operatorname{Hom}_R(B, C) \arrow[r] \arrow[d, "\psi_C"] & 0 \\
0 \arrow[r] & \operatorname{Hom}_R(\Pi/B, C) \arrow[r] & \operatorname{Hom}_R(\Pi, C) \arrow[r, "\tau"] & G(C) \arrow[r] & 0
\end{tikzcd}$$

To see why the top row is exact, recall that C is injective so that $\operatorname{Hom}_R(-, C)$ is an exact functor. For the bottom row, recall that τ is surjective, and the previous claim established that $\operatorname{Ker}\tau = \operatorname{Ker} j^*$. This immediately gives that $\operatorname{Coker}\tau = \operatorname{Coker} j^*$, and,

$$\psi_C : \operatorname{Hom}_R(B, C) \to G(C)$$
$$f \mapsto (Gf)(\theta e)$$

is an isomorphism (this is a consequence of the Snake Lemma). Consider the map

$$M \to C^{\operatorname{Hom}_R(M,C)}$$
$$m \mapsto (f(m))_{m\in M}$$

As C is a cogenerator, the above is an injective map. Let N be the cokernel of the above; there is an injection $N \hookrightarrow C^Y$ for some set Y. We can construct the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & C^{\operatorname{Hom}_R(M,C)} \arrow[r, dashed] \arrow[dr] & C^Y \\
 & & & N \arrow[u]
\end{tikzcd}$$

and, applying the functors G and $\operatorname{Hom}_R(B, -)$, we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_R(B, M) \arrow[r] \arrow[d, "\psi_M"] & \operatorname{Hom}_R(B, C^{\operatorname{Hom}_R(B,C)}) \arrow[r] \arrow[d, "\psi_{C^{\operatorname{Hom}_R(M,C)}}"] & \operatorname{Hom}_R(B, C^Y) \arrow[d, "\psi_{C^Y}"] \\
0 \arrow[r] & G(M) \arrow[r] & G(C^{\operatorname{Hom}_R(M,C)}) \arrow[r] & G(C^Y)
\end{tikzcd}$$

Since G and $\mathrm{Hom}_R(B,-)$ preserve limits,

$$\mathrm{Hom}_R(B, C^{\mathrm{Hom}_R(B,C)}) = \mathrm{Hom}_R(B,C)^{\mathrm{Hom}_R(B,C)}$$

$$G(C^{\mathrm{Hom}_R(M,C)}) = G(C)^{\mathrm{Hom}_R(M,C)}$$

and likewise for $\mathrm{Hom}_R(B, C^Y)$ and $G(C^Y)$. This implies that

$$\psi_{C^{\mathrm{Hom}_R(M,C)}}, \quad \psi_{C^Y}$$

are isomorphisms. By the Five Lemma, ψ_M must also be an isomorphism. It remains only to prove naturality of the association $\psi \mapsto \psi_M$; that is, the following diagram commutes for $f \in \mathrm{Hom}_R(M,N)$:

$$\begin{tikzcd} \mathrm{Hom}_R(B,M) \arrow[r, "\psi_M"] \arrow[d, "f_*"] & G(M) \arrow[d, "Gf"] \\ \mathrm{Hom}_R(B,N) \arrow[r, "\psi_N"] & G(N) \end{tikzcd}$$

Let $\phi \in \mathrm{Hom}_R(B,M)$:

$$\begin{aligned} (Gf)(\psi_M(\phi)) &= (Gf)(G\phi)(\theta e) \\ &= (G(f\phi))(\theta e) \\ &= (G(f_*\phi))(\theta e) \\ &= \psi_N(f_*\phi) \end{aligned}$$

Whence naturality follows, and the theorem is proved. $\square$

Using all of the above theorems, we get some quick corollaries:

Corollary 1.21. *If $F : R - \boldsymbol{Mod} \to \boldsymbol{Ab}$ is an additive functor, then the following are equivalent:*

(1) *F preserves direct limits.*

(2) *F is right exact and preserves direct sums.*

(3) *$F \cong - \otimes_R B$ for some left R-module B.*

(4) *F has a right adjoint; that is, there is a functor $G : Ab \to \boldsymbol{Mod}_R$ such that (F,G) is an adjoint pair.*

Corollary 1.22. *If $G : R - \textbf{\textit{Mod}} \to \textbf{\textit{Ab}}$ is an additive functor, then the following are equivalent:*

(1) *G preserves inverse limits.*
(2) *G is left exact and preserves direct products.*
(3) *G is representable; that is, $G \cong \mathrm{Hom}_R(B, -)$ for some left R module B.*
(4) *G has a left adjoint; that is, there is a functor $F : \textbf{\textit{Ab}} \to \textbf{\textit{Mod}}_R$ such that (F, G) is an adjoint pair.*

2. Balancing Ext and Tor

The idea of balancing Ext and Tor comes from the observation that there are two possible ways to compute $\mathrm{Ext}^i_R(A, B)$ (and similarly for $\mathrm{Tor}^R_i(A, B)$.

Firstly, one may choose a projective resolution $F_i \to A \to 0$. Applying the functor $\mathrm{Hom}_R(-, B)$, the homology groups (modules) of the resulting sequence yields $\mathrm{Ext}^i_R(A, B)$.

However, we may just as easily choose an injective resolution $0 \to B \to Q_i$ and apply the functor $\mathrm{Hom} - R(A, -)$. Again, the homology of the resulting sequence yields $\mathrm{Ext}^i_R(A, B)$. If these definitions were well defined, one should expect that these processes yield the same answer in both cases.

We first need the following definition (in order to try to keep the notes somewhat self contained, I've included this, but Rob's notes covered this material):

Definition 2.1. Let $\mathcal{C}$ be a double complex. We define the total complexes

$$\mathrm{Tot}^{\Pi}(\mathcal{C}) \quad \mathrm{Tot}^{\oplus}(\mathcal{C})$$

such that

$$\mathrm{Tot}^{\Pi}(\mathcal{C})_n = \prod_{p+q=n} C_{pq} \quad \mathrm{Tot}^{\oplus}(\mathcal{C}) = \bigoplus_{p+q=n} C_{pq}$$

with differentials

$$d_n^{\Pi} = \prod_{p+q=n} d_{pq}^h + \prod_{p+q=n} d_{pq}^v$$
$$d_n^{\oplus} = \bigoplus_{p+q=n} d_{pq}^h + \bigoplus_{p+q=n} d_{pq}^v$$

Definition 2.2. Let P and Q be chain complexes of right and left R-modules, respectively. Form the double complex $P \otimes_R Q = \{P_p \otimes_R Q_q\}$ with horizontal differentials $d \otimes 1$ and vertical differentials $(-1)^p \otimes d$.

We define the **total tensor product complex** to be

$$\mathrm{Tot}^{\oplus}(P \otimes_R Q)$$

Definition 2.3. Given a chain complex P and a cochain complex I, we may form the double cochain complex $\mathrm{Hom}(P, I) = \{\mathrm{Hom}(P_p, I^q)$.

We have horizontal differentials d^h such that $(d^h f)(p) = f(dp)$ for $f \in \mathrm{Hom}(P_p, I_q$, and vertical differentials

$$(d^v f)(p) = (-1)^{p+q+1} d(fp)$$

Then, $\mathrm{Hom}(P, I)$ is called the Hom double complex, and,

$$\mathrm{Tot}^{\Pi}(\mathrm{Hom}(P, I))$$

is called the **total Hom cochain complex**.

We will need the following:

Lemma 2.4 (Acyclic Assembly Lemma)**.** *Let C be a double complex in $\boldsymbol{Mod}_R$. Then,*

- *$Tot^{\Pi}(C)$ is an acyclic chain complex, assuming either of the following:*

(1) *C is an upper half plane complex with exact columns.*

(2) *C is a right half plane complex with exact rows.*

- *$Tot^{\oplus}(C)$ is an acyclic chain complex, assuming either one of the following:*

(1) *C is an upper half plane complex with exact rows.*

(2) *C is a right half plane complex with exact columns.*

Proof. We will prove (1), and then justify why (1) in fact implies every other case. Recall that d^v and d^h will denote our vertical and horizontal differentials, respectively. Let $(\dots, c_{-pp}, \dots, c_{-22}, c_{-11}) \in \mathrm{Tot}^{\Pi}(C)$; in particular, it should be an element of the kernel of our total chain complex diferentials.

We now describe an iterative process to find the preimage of the above tuple.

Step 0: Choose $b_{10} = 0$. Since we are at the bottom of our double complex, the vertical differentials are all surjective so that we way choose $b_{01} \in C_{01}$ such that $d^v(b_{01}) = c_{00}$.

Step 1: By definition of our total complex differentials, we know that since $(\dots, c_{-pp}, \dots, c_{-22}, c_{-11}) \in \operatorname{Ker} d$,

$$d^h(c_{00}) + d^v(c_{-11}) = 0$$

By step 0, $d^v(b_{01}) = c_{00}$, and, recalling that double chain complexes are anticommutative,

$$\begin{aligned} d^h(c_{00}) + d^v(c_{-11}) &= d^h(d^v(b_{01})) + d^v(c_{-11}) \\ &= d^v(c_{-11}) - d^v d^h(b_{01}) = 0 \end{aligned}$$

Whence $c_{-11} - d^h(b_{01}) \in \operatorname{Ker} d^v$, and by exactness of our columns we may find b_{-12} such that $d^v(b_{-12}) = c_{-11} - d^h(b_{01})$, so

$$d^v(b_{-12}) + d^h(b_{01}) = c_{-11}$$

Step k: Assume that we have successfully chosen our b_{ij} for the previous iterations. Then, again since our tuple is in the kernel of our total complex differential,

$$d^v(c_{-kk}) + d^h(c_{-k+1,k-1}) = 0$$

By assumption we have already chosen $b_{-k+1,k}$, $b_{-k+2,k-1}$ such that

$$d^v(b_{-k+1,k}) + d^h(b_{-k+2,k-1}) = c_{-k+1,k-1}$$

so that

$$\begin{aligned} d^v(c_{-kk}) + d^h(c_{-k+1,k-1}) &= d^v(c_{-kk}) + d^h(d^v(b_{-k+1,k}) + d^h(b_{-k+2,k-1})) \\ &= d^v(c_{-kk}) - d^v d^h(b_{-k+1,k}) + d^h d^h(b_{-k+2,k-1}) \\ &= d^v\big(c_{-kk} - d^h(b_{-k+1,k})\big) = 0 \end{aligned}$$

And by exactness of our columns, we find $b_{-k,k+1}$ such that

$$d^v(b_{-k,k+1}) = c_{-kk} - d^h(b_{-k+1,k})$$

So $c_{-kk} = d^v(b_{-k,k+1}) + d^h(b_{-k+1,k})$.

By induction, we iterate this process indefinitely; the resulting element of $\operatorname{Tot}^{\Pi}(C)_1$ is by construction the preimage of the given element in $\operatorname{Tot}^{\Pi}(C)_0$. Now, one will notice that the above algorithm is in no way dependent upon the base point. Hence we may apply the above to $\operatorname{Tot}^{\Pi}(C)_n$ for every n to deduce that $\operatorname{Tot}^{\Pi}(C)$ is acyclic, as desired.

Now, to deduce (2) from (1), merely transpose the order of the indices in the above process; this immediately gives acyclicity. Similarly, if we assume (4) true, we again just transpose in order to deduce the case of (3). It remains to show (1) $\implies$ (4).

To see this explicitly, recall that we are working with left R-modules. Given a right half plane double complex C, we truncate (non-stupidly) the vertical diferentials at the nth step. Then, in the resulting total complex, the every component will only consist of a finite direct sum of terms. In the finite case, of course, the direct sum and direct product coincide so we may apply (1) to deduce that the truncated complex $\mathrm{Tot}^{\oplus}(\tau_n C)$ is acyclic. As n is arbitrary, we deduce that $\mathrm{Tot}^{\oplus}(C)$ remains acyclic as well. This completes the proof.

□

Using the above, we may answer the questions posed at the beginning of this section in the affirmative:

Theorem 2.5. *For any right R-module A and left R-module B,*

$$L_n(A\otimes_R)(B) = R_n(\otimes_R B)(A) = \mathrm{Tor}_i^R(A, B)$$

Proof. Observe first that if we consider A and B as chain complexes concentrated in degree 0, then the total tensor product with any other chain complex C is merely the chain complex with $A \otimes_R C_n$ (resp. $C_n \otimes_R B$) in the nth spot.

Choose projective resolutions $\epsilon : P \to A$, $\eta : Q \to B$ and consider the 3 tensor product complexes $P \otimes Q$, $A \otimes Q$, and $P \otimes B$.

We may augment $P \otimes Q$ along the left with $A \otimes Q$, and along the bottom with $P \otimes B$. Taking the total tensor product, we have

$$A \otimes Q \xleftarrow{\epsilon\otimes Q} \mathrm{Tot}(P \otimes Q) \xrightarrow{P\otimes\eta} P \otimes B$$

Translating the augmented double complex to the left and taking the total complex induces the mapping cone of $\epsilon \otimes Q$; recall then that cone(f) is exact if and only if f is a quasi-isomorphism (cf. my previous notes).

Note however that the functor $\otimes_R Q$ is exact since Q is projective, and every projective module is flat. Hence the translated double complex has exact rows; employing the Acyclic Assembly Lemma, we deduce that cone$(\epsilon \otimes Q)$ is acylic, so that $\epsilon \times Q$ is a quasi-isomorphism.

Augmenting along the bottom and taking the total complex, we again merely get the mapping cone of $P \otimes \eta$. By identical reasoning as above, this is also a quasi-isomorphism. Whence

$$L_n(A\otimes_R)(B) = H_n\big(\mathrm{Tot}(P \otimes Q)\big) = R_n(\otimes_R B)(A)$$

And by transitivity, the result is immediate.

□

Theorem 2.6. *For every pair of R-modules A and B,*

$$R^n \operatorname{Hom}_R(A, -)(B) = R^n \operatorname{Hom}_R(-, B)(A) = \operatorname{Ext}^i_R(A, B)$$

The proof of the above is nearly identical to the Tor case; we consider augmenting our double complex and shifting. The total complex of this shifted double complex induces the mapping cone of the induced morphisms and by projectivity/injectivity, our functors are exact. We conclude by employing the Acyclic Assembly Lemma again.