Integragtion Basics (152)

Exponential and Logarithmic Functions

Basics

$0 < b \neq 1$ and recall $\ln \equiv \log_e$

$$f(x) = b^x \equiv e^{x\ln(b)} \qquad \text{Domain: } (-\infty, \infty) \text{ Range: } (0, \infty)$$

$$g(x) = \log_b(x) \text{ inverse of } f(x) = b^x \qquad \text{Domain: } (0, \infty) \text{ Range: } (-\infty, \infty)$$

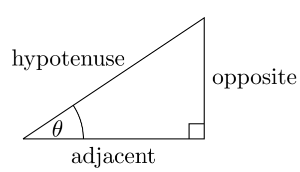
$$y = \log_b(x) \iff x = b^y$$

$$(\log_a(b)) \cdot (\log_b(c)) = \log_a(c) \implies \log_a(x) = \frac{\ln(x)}{\ln(a)}$$

$x, y > 0$ & $r \in \mathbb{R}$	$x, y > 0$ & $r \in \mathbb{R}$
$b^{\log_b(x)} = x$	$\log_b(b^x) = x$
$\log_b(1) = 0$	$b^0 = 1$
$\log_b(xy) = \log_b(x) + \log_b(y)$	$b^x \cdot b^y = b^{x+y}$
$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$	$\frac{b^x}{b^y} = b^{x-y}$
$\log_b(x^r) = r \cdot (\log_b(x))$	$(b^x)^r = b^{xr}$
	$(ab)^x = a^x \cdot b^x$ and $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

Basic Trig

Basics



$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} \quad \sin(\theta) = \frac{\text{opp}}{\text{hyp}} \quad \tan(\theta) = \frac{\text{opp}}{\text{adj}}$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \quad \sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} \quad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

Inverse Trig

Basics

$$y = \sin(\theta) \implies \theta = \sin^{-1}(y) \quad \text{where} \quad -1 \le y \le 1 \quad \text{and} \quad \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$y = \cos(\theta) \implies \theta = \cos^{-1}(y) \quad \text{where} \quad -1 \le y \le 1 \quad \text{and} \quad 0 \le \theta \le \pi$$

$$y = \tan(\theta) \implies \theta = \tan^{-1}(y) \quad \text{where} \quad y \in \mathbb{R} \quad \text{and} \quad \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$y = \cot(\theta) \implies \theta = \cot^{-1}(y) \quad \text{where} \quad y \in \mathbb{R} \quad \text{and} \quad 0 \le \theta \le \pi$$

$$y = \sec(\theta) \implies \theta = \sec^{-1}(y) \quad \text{where} \quad |y| \ge 1 \quad \text{and} \quad 0 \le \theta \le \pi,\ \theta \ne \frac{\pi}{2}$$

$$y = \csc(\theta) \implies \theta = \csc^{-1}(y) \quad \text{where} \quad |y| \ge 1 \quad \text{and} \quad \frac{-\pi}{2} \le \theta \le \frac{\pi}{2},\ \theta \ne 0$$

Fundamental Theorem of Calculus

Let $f : [x, b] \to \mathbb{R}$ be a continuous function.
Let $F : [a, b] \to \mathbb{R}$ be a function.

- if F is an antiderivative of f on $[a, b]$ (i.e. $F'(x) = f(x)$ for each $x \in [a, b]$), then

$$\int_a^b f(x)dx \equiv \int_a^b F'(x)dx = F(b) - F(a)$$

- if $F(x) = \int_a^x f(t)dt$ for each $x \in [a, b]$, then F is an anti-derivative of f on $[a, b]$, i.e.

$$F'(x) \equiv D_x\left[\int_a^x f(t)dt\right] = f'(t)$$

Differentiation Rules

Basics

Let $y = f(x)$ and $y = g(x)$ be functions which are differentiable at x. Let a and b be constants.

$$D_x\left[af(x) + bg(x)\right] = af'(x) + bg'(x)$$

$$D_x\left[f(x) \cdot g(x)\right] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

In the case $g(x) \ne 0$

$$D_x\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

In the case that f is diferentiable at x and g is differentiable at $f(x)$

$$D_x\left[g\Big(f(x)\Big)\right] = g'\Big(f(x)\Big)f'(x)$$

Integral Formulas

Derivatives	$\overset{\text{FTC}}{\Longrightarrow}$	Integrals
$D_x[u^n] = nu^{n-1}\dfrac{du}{dx}$		$\int u^n du \overset{n\neq -1}{=} \dfrac{u^{n+1}}{n+1} + C$
$D_x e^u = e^u \dfrac{du}{dx}$		$\int e^u du = e^u + C$
$D_x \ln\lvert u\rvert \overset{u\neq 0}{=} \frac{1}{u}\frac{du}{dx}$		$\int \frac{du}{u} \overset{u\neq 0}{=} \ln\lvert u\rvert + C$
$D_x a^u = a^u \ln a \frac{du}{dx}$	$0 < a \neq 1$	$\int a^u\, du = \frac{a^u}{\ln a} + C$
$D_x \sin u = \cos u \frac{du}{dx}$		$\int \cos u\, du = \sin u + C$
$D_x \tan u = \sec^2 u \frac{du}{dx}$		$\int \sec^2 u\, du = \tan u + C$
$D_x \sec u = \sec u\ \tan u\ \frac{du}{dx}$		$\int \sec u\ \tan u\, du = \sec u + C$
$D_x \cos u = -\sin u \frac{du}{dx}$		$\int \sin u\, du = -\cos u + C$
$D_x \cot u = -\csc^2 u \frac{du}{dx}$		$\int \csc^2 u\, du = -\cot u + C$
$D_x \csc u = -\csc u\ \cot u \frac{du}{dx}$		$\int \csc u\ \cot u\, du = -\csc u + C$
$D_x \sin^{-1} u = \dfrac{1}{\sqrt{1-u^2}}\frac{du}{dx} = -D_x \cos^{-1} u$		$\int \dfrac{du}{\sqrt{a^2-u^2}} \overset{a\geq 0}{=} \sin^{-1}\frac{u}{a} + C$
$D_x \tan^{-1} u = \dfrac{1}{1+u^2}\frac{du}{dx} = -D_x \cot^{-1} u$		$\int \dfrac{du}{a^2+u^2} \overset{a\geq 0}{=} \frac{1}{a}\tan^{-1}\frac{u}{a} + C$
$D_x \sec^{-1} u = \dfrac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx} = -D_x \csc^{-1} u$		$\int \dfrac{du}{u\sqrt{u^2-a^2}} \overset{a\geq 0}{=} \frac{1}{a}\sec^{-1}\frac{\lvert u\rvert}{a} + C$

More Integrals

$\int \tan u\, du$	$= -\ln\lvert\cos u\rvert + C$	$= \ln\lvert\sec u\rvert + C$
$\int \cot u\, du$	$= \ln\lvert\sin u\rvert + C$	$= -\ln\lvert\csc u\rvert + C$
$\int \sec u\, du$	$= \ln\lvert\sec u + \tan u\rvert + C$	$= -\ln\lvert\sec u - \tan u\rvert + C$
$\int \csc u\, du$	$= -\ln\lvert\csc u + \cot u\rvert + C$	$= \ln\lvert\csc u - \cot u\rvert + C$

Integration by Parts

$$\int u\, dv = uv - \int v\, du$$

Key Ideas in Integration by Parts.

- For $\int x^n f(x)\, dx$ where $\int f(x)\, dx$ is easy, try $u = x^n$ and $dv = f(x)\, dx$. (Note that then $v = \int dv = \int f(x)\, dx$) This often reduces x^n to x^{n-1}.
- For $\int f(x)\, dx$ if the integrand $f(x)$ is easy to differentiate but hard to integrate, then try letting $u = f(x)$ and so $dv = dx$.
- Bring to the other side (i.e. loops) method.
- Creatively look for a dv that is easy to integrate (since $v = \int dv$).

Trig Identities useful in Integration

Half-Angle Formulas:	$\cos^2 x = \dfrac{1+\cos(2x)}{2}$	$\sin^2 x = \dfrac{1-\cos(2x)}{2}$
Double-Angle Formulas:	$\cos(2x) = \cos^2 x - \sin^2 x$	$\sin(2x) = 2\sin x\ \cos x$

Add./Subst. Formulas:

$$\cos(s+t) = \cos s\ \cos t - \sin s\ \sin t$$
$$\sin(s+t) = \sin s\ \cos t + \cos s\ \sin t$$
$$\cos(s-t) = \cos s\ \cos t + \sin s\ \sin t$$
$$\sin(s-t) = \sin s\ \cos t - \cos s\ \sin t$$

Trig Substitution

If Integrand Involves	Then Make the Substitution	Restriction on θ
$a^2 - u^2$	$u = a\sin\theta \leftrightsquigarrow \theta = \sin^{-1}\frac{u}{a}$	$\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$a^2 + u^2$	$u = a\tan\theta \leftrightsquigarrow \theta = \tan^{-1}\frac{u}{a}$	$\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$u^2 - a^2$	$u = a\sec\theta \leftrightsquigarrow \theta = \sec^{-1}\frac{u}{a}$	$0 \leq \theta \leq \pi,\ \theta \neq \frac{\pi}{2}$

Commonly Occurring Limits

- $\lim_{x\to\infty} \dfrac{\ln(x)}{x} = 0$
- $\lim_{x\to\infty} c^{\frac{1}{x}} = 1 \quad (c > 0)$
- $\lim_{x\to\infty} c^x = 0 \quad (|c| < 1)$

Indeterminate forms

Many times when working limits we want to write indeterminate forms like:

$$\frac{0}{0} \quad \frac{\pm\infty}{\pm\infty} \quad 0\cdot(\pm\infty) \quad 1^{\infty} \quad 0^0 \quad \infty^0 \quad \infty - \infty$$

L'Hopital's

Basics: Suppose that we have one of the following cases,

$$\lim_{x\to a} \frac{f(x)}{g(x)} \text{``}=\text{''} \frac{0}{0} \qquad \textbf{OR} \qquad \lim_{x\to a} \frac{f(x)}{g(x)} \text{``}=\text{''} \frac{\pm\infty}{\pm\infty}$$

where a can be any real number, infinity or negative infinity. In these cases we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

L'Hopital's for $0 \cdot (\pm\infty)$

Suppose that we have,

$$\lim_{x\to a} f(x) \cdot g(x) \text{``}=\text{''}\ 0 \cdot (\pm\infty)$$

Then we can write as either

$$f(x) \cdot g(x) = \frac{f(x)}{\left(\frac{1}{g(x)}\right)} \qquad \textbf{OR} \qquad f(x) \cdot g(x) = \frac{g(x)}{\left(\frac{1}{f(x)}\right)}$$

and next try to use L'Hopitals

L'Hopital's for other indeterminate forms

Suppose that we have,

$$\lim_{x\to a} f(x)^{g(x)} \text{``}=\text{''}\ 1^{\infty} \qquad \textbf{OR} \qquad \lim_{x\to a} f(x)^{g(x)} \text{``}=\text{''}\ 0^{0} \qquad \textbf{OR} \qquad \lim_{x\to a} f(x)^{g(x)} \text{``}=\text{''}\ \infty^{0}$$

Then we can write

$$\ln\left(f(x)^{g(x)}\right) = g(x) \cdot \ln\left(f(x)\right) \qquad \textbf{AND} \qquad f(x)^{g(x)} = e^{\ln\left(f(x)^{g(x)}\right)}$$

AND SO

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{\ln\left(f(x)^{g(x)}\right)} = e^{\lim_{x\to a} \ln\left(f(x)^{g(x)}\right)}$$

Now notice that

$$\lim_{x\to a} \ln\left(f(x)^{g(x)}\right) = \lim_{x\to a} g(x) \cdot \ln\left(f(x)\right)$$

is now in one of the previous indeterminate forms!