# Outline:

**The Slogan:** Hard Question: "when are two varieties Derived equivalent?" (given $X$ & $Y$ (var.) are they derived equivalent)

Where to start??? How about where someone has already guessed...

[ Conjecture [Bondal] Two Complex projective varieties related by a flop are derived equivalent ]

[BDF] has a conjectural solution

- ✓ The Example:
  - Grassmann ✓ flop / Atiyah ✓ flop
- ✓ The Window Approach (brief sketch)
- Kernel Approach:
  - ✓ The Benifits of kernels (brief)
  - ✓ kernels for Action
    - ✓ Algebraic
    - ✓ partial compactifications
    - ✓ Drinfield
  - kernels for windows
    - ✓ Bousfield
  - Resolutions for generators [Kapranov Connection]
    - ✓ Motivation by Resolution of the diagonal
      - kemf method
- ✓ Conclusion:
  "So what should be the kernel program?"

# Kernels for Grassmann Flops

[Joint work w/ Ballard, Chidambaram, Favero, McFaddin]

✱ don't forget to start w/ slogan/ conjecture ✱

---

**Notation:** $V$ & $W$ fin. dim. Vect. Sp. over $\mathbb{C}$ w/ $\dim V = d_V$ & $\dim W = d_W$ $d_W > d_V$ $\operatorname{spec}(\mathbb{C}) = \{*\}$

---

**Grassmann Flop:** [introduced by Don & Seg]

$$Z := \operatorname{Tot}\left(\operatorname{Hom}(V,W) \oplus \operatorname{Hom}(W,V)\right)$$

$$GL(V) \circlearrowleft Z$$

[the obvious action realizing $\operatorname{Hom}(W,V) = \left(\operatorname{Hom}(V,W)\right)^\vee$]

D-M stack

$$Z_0 := \left[Z / GL(V)\right]$$

∃ two (Mumford) G.I.T. Quotients

(i.e. symplectic reductions)

momentmaps:

$(A,B) \longmapsto i\left(A^*A - \mathbb{1}, \ \mathbb{1}\right)$

$(A,B) \longmapsto i\left(\mathbb{1}, \ B^*B - \mathbb{1}\right)$

$$\begin{tikzcd}
 & Z_0 & \\
Z_+ \arrow[ur, "\text{open}"] \arrow[rr, dashed, leftrightarrow, "\text{bi-rational}"] & & Z_- \arrow[ul, "\text{open}"']
\end{tikzcd}$$

here I mean total space again

i.e. full rank!

denote $\mathcal{U}_+$

$$\left[\frac{\{\phi \mid \operatorname{rk}\phi = d_V\} \oplus \operatorname{Hom}(W,V)}{GL(V)}\right] \cong Z_+$$

total space

i.e. full rank!

$$Z_- = \left[\frac{\operatorname{Hom}(V,W) \oplus \{\phi \mid \operatorname{rk}\phi = d_V\}}{GL(V)}\right]$$

denote $\mathcal{U}_-$

denote $S$ as the tautological bundle of $Gr(d_V, W)$ then $Z_+ \cong \operatorname{Tot}\left(S^{d_W}\right)$

Both representable as vector bundles over the Grassmannians

or a fiber description:

$$\begin{tikzcd}
Gr(d_V,W) \arrow[d, "t"'] & Z_+ \\
\Bbbk & \operatorname{Hom}(W,V) \arrow[u, mapsto, "\operatorname{Tot}\circ t^*"']
\end{tikzcd}$$

fiber description:

$$\begin{tikzcd}
Gr(d_W - d_V,W) \arrow[d, "\underset{\sim}{t}"'] & Z_- \\
\Bbbk & \operatorname{Hom}(V,W) \arrow[u, mapsto, "\operatorname{Tot}\circ \underset{\sim}{t}^*"']
\end{tikzcd}$$

✱ make sure to mention ✱

**Three-fold flop:** (example of Atiyah flop)

$d_V = 1$, $d_W = 2$

then

$Z = W \oplus W^\vee$ $GL(V) = \mathbb{G}_m$

and

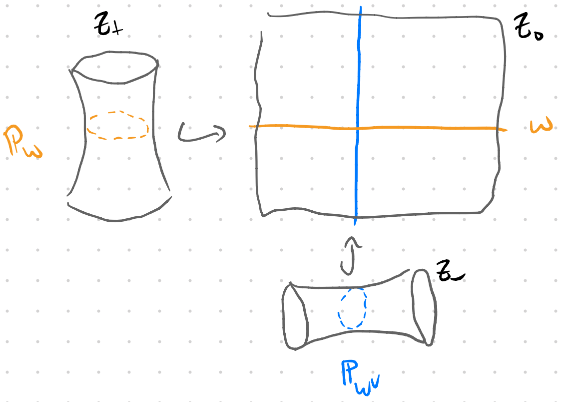
$$Z_+ = \left[\frac{(W \setminus \{0\}) \oplus W^\vee}{\mathbb{G}_m}\right] \cong \operatorname{Tot}\left(\mathcal{O}_{\mathbb{P}_W}(-1)^{\oplus 2}\right)$$

$$Z_- = \left[\frac{W \oplus (W^\vee \setminus \{0\})}{\mathbb{G}_m}\right] \cong \operatorname{Tot}\left(\mathcal{O}_{\mathbb{P}_{W^\vee}}(-1)^{\oplus 2}\right)$$

For any Atiyah flop Bondal & Orlov showed $X_+$ & $X_-$ are derived equivalent, and provided a kernel

✱ Note $Z_+$ & $Z_-$ are Calabi-Yau 3-folds

picture to keep in mind...



Don & Seg's idea is to take advantage of this inclusion in the respected derived categories

**Previous Results**: (for general grass. flop) [Donovan & segal]

$D^b(X_+) \cong D^b(X_-)$ Via "Windows"

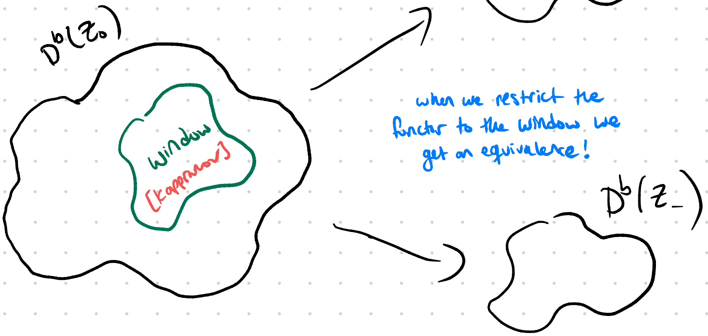
[did not provide a kernel]

↑ for the general...

*say derived categories of coherent sheaves...

## Idea of Windows:

$$\begin{tikzcd} & D^b(Z_+) \\ D^b(Z_0) \arrow[ur, "\text{derived restriction}"] \arrow[r, "\text{derived Restriction}"'] & D^b(Z^-) \end{tikzcd}$$



when we restrict the functor to the window we get an equivalence!

Answer to:

aren't these already isomorphic?

→well yes

But what we want to understand a more general phenomia of "flops"

**Conjecture**: [Bondal]

Two proj. Var. related by a flop should be derived equivalent

i.e. the derived category is an invariant of flops

**flops**: "essentially" the Morphism

$$\begin{tikzcd} & Y \arrow[dl] \arrow[d, "f"] \\ Y^+ \arrow[r, dashed, "f^+"] & X \end{tikzcd}$$

small contraction

-D (a/any) f-ample

$D^+$ $f^+$-ample

D in $Y^+$

$Y \to X$

$Y^+ \to X$

D- flop

in our example

$Y = Z_+$ $X = Z_-$ $Y^+ = Z_-$

## example of windows:

3-fold flop:

$\mathcal{W} = \langle \mathcal{O}, \mathcal{O}(1) \rangle$

general grass. flop:

$\mathcal{K} =$ full triangulated subcategory generated by GL(W)-irreps represented by Young diagrams of height $\le d_W - d_V$ & width $\le d_V$

& $\iota_\pm : \mathcal{W} \to Z_\pm$ * are equivalences

* and spherical functors *

---

unfortunately Don & seg did not provide a kernel for this equivalence

The "point" of this work is to provide a kernel which will "make" the window!

## ★ Benifits of kernels:

- understanding how these equivalences behave under base change [using a result of Bondal about decent]

* Allows us to extend results to other fields in this example to number/function fields

**Kernels**

(do I need to?)

$K \in D^b(X \times Y)$

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

$\Phi_K : D(X) \longrightarrow D(Y)$

$M \in D(X)$

$$\Phi_K(M) := R(\pi_Y)_*\left(K \overset{L}{\otimes} L(\pi_X)^* M\right)$$

Integral transform...

**The kernel:** generalization of the kernel in Ballard, Diemer, Favero [BDF] shown to realize the derived equivalence of the Atiyah Flop...

Bondal & Orlov gave kernel for Atiyah flop: $U_+ \times_{Z//_0} U_-$

affine Quotient

**spoiler:** that's the kernel in the general grass flop as well

## Kernels for an Action: (wall crossing "first")

*"duality" of the action*

**Algebraically:** $R := H^0(Z, \mathcal{O}_Z)$

$$GL(W) \times Z \overset{\sigma}{\underset{\pi}{\rightrightarrows}} Z$$

(action) (projection)

$$Q := \left(\sigma^*(R), \pi^*(R), \mathbb{C}[GL(W)]\right) \subseteq \mathbb{C}[GL(W) \times Z]$$

$R \cong \mathbb{C}[A, B]$ A → matrix of indeterminates $d_V \times d_W$

B → " $d_W \times d_V$

(i.e. the "rights" (R) and "lefts" (L) are $\sigma^\#(R)$ and $p^\#(R)$)

$$Q \cong \mathbb{C}[A^L, A^R, B^L, B^R, C] / \langle B^L - B^R C, A^R - CA^L \rangle$$

C → matrix of indet. $d_W \times d_W$

2. ways to find semi-stable:

1. linearized line bun.
2. Hilbert-Mumford numerical criterion

here have to try a lot of characters...

difficult

* finding ^ semi-stable loci are in general not super easy as they depend on linearized line bundles and many give same semi-stable, actually this "equivalence" defines convex hulls in the ample cone of linearized line bundles ⊗Q

but

**Geometrically:**

**partial compactification** (of the GL(W)-action)

$$\begin{tikzcd} & \mathrm{Spec}(Q) \arrow[d, shift right, "s"'] \arrow[d, shift left, "p"] \\ GL(W) \times Z \arrow[ur, "i", "\text{inclusion}"'] \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\pi"'] & Z \end{tikzcd}$$

(s rest. to σ & p rest. to π)

why is this helpful... well just as any other compactification we have the concept of boundary

$$\partial_{\mathrm{Spec}(Q)} := \mathrm{Spec}(Q) \setminus i(GL(W) \times Z)$$

[i.e. the additional points]

**Result:**

$$Z_s^{ss} := Z \setminus s(\partial)$$

$$Z_p^{ss} := Z \setminus p(\partial)$$

and these are exactly the semi-stable loci that give us our two G.I.T. Quotients as before

i.e. $Z_+ = \left[Z_s^{ss}/GL(W)\right]$ & $Z_- = \left[Z_p^{ss}/GL(W)\right]$

# Kernels for an Action: (continued...)

Don't forget!

**Drinfeld:** (A curious connection) *[BDF conjectures a kernel created by this construction should provide a solution to Bondal's Conjecture]*

Idea: you want to know what's happening look @ the "points"

originally drinfeld used a similar construction for $\mathbb{G}_m$ and in BDF they showed the two constructions aligned...

First capture "Duality" of the action

*again I mean total space over spec($\mathbb{C}$)*

$$\mathcal{X} := \mathrm{End}(V) \oplus (\mathrm{End}(V))^{\vee} \longrightarrow \mathrm{End}(V)$$

$GL(V) \circlearrowleft$ (mult. on left)

$GL(V) \circlearrowleft$ (mult right by inverse)

Trivial $GL(V)$-action

*captures the prototypical $GL(V)$ invariance*

$$(\Phi, \Psi) \longmapsto \Psi \cdot \Phi$$

*Note $GL(V)$-equivariant*

[Note: $\mathcal{Z} \to \mathrm{End}(V)$

$(\alpha, \beta) \mapsto \beta \circ \alpha$ (as well :)]

for a scheme $S$ over $\mathrm{End}(V)$ denote $\mathcal{X}_S := \mathcal{X} \times_{\mathrm{End}(V)} S$

as map to End(V) is GL(V)-invariant $\mathcal{X}_S$ naturally inherits an action of GL(V)

then define $d(\mathcal{Z})$

$$\mathrm{Hom}_{\mathrm{End}(V)}(S, d(\mathcal{Z})) := \mathrm{Hom}_{GL(V)}(\mathcal{X}_S, \mathcal{Z})$$

GL(V) equivariant Hom's

$$\boxed{\text{Thm: } d(S) \text{ is a scheme } \& \ d(\mathcal{Z}) = \mathrm{Spec}(Q)}$$

---

**So in 3-fold case:** $(d_V = 1)$

"$\mathrm{End}(V)$" = "$\mathbb{C}^1$"

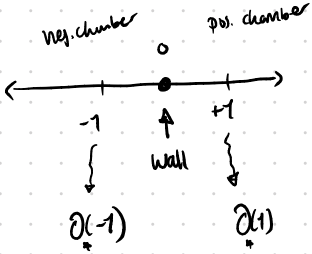
i.e.

$$\mathcal{X} = \mathbb{A}^1_+ \times \mathbb{A}^1_-$$

positive grading

negative grading

Moduli space of characters ≅ moduli space of $\mathbb{G}_m$-linearized linebundles



## So what's going on? (no a serious question)

### $\mathbb{G}_m$-case:

**one idea:** $\mathbb{A}^1$ is the "natural" monoidal compactification of $\mathbb{G}_m$

**another idea:** $\mathbb{A}^1_+$ & $\mathbb{A}^1_-$ are rk 1 (geo) bundles over $\mathrm{spec}(\mathbb{C}) := \{*\}$

only (up to iso) irrep of $\mathbb{G}_m$

associated to the rk 1 free sheaves which represent the two "non-trivial" chambers of $\mathbb{G}_m$...

*perhaps only rank 1 in this case since all irreps of $\mathbb{G}_m$ are 1-dim*

### GL(V)-case:

ditto

*not rk 1 over spec($\mathbb{C}$)!*

*here irreps are not only 1-dim*

yet $k[\mathrm{End}(V)] = \bigoplus_{\substack{V_i \\ \text{polynomial irrep}}} V_i$ and $k[(\mathrm{End}(V)^{\vee})] = \bigoplus_{\substack{V_i^{\vee} \\ \text{polynomial irrep}}} V_i$

So they somehow make up all the possible rk 1 bundles over all possible irreps...

*what is $S$ doing?* Somehow collecting the "data" of trivial action...

i.e. the part that GL acts trivially on can act "wildly"

Cheat Sheet
for semi-orth

$\times\!\!\to$

$\leftarrow$ ✓

## Bousfield: (continued...)

Lem] let $C_1 \xrightarrow{\varepsilon_1} \mathbb{1} \xrightarrow{\delta_1} L_1$ and $C_2 \xrightarrow{\varepsilon_2} \mathbb{1} \xrightarrow{\delta_2} L_2$ be Bousfield triangles s/t $L_1 C_1 \xrightarrow{L_1(\varepsilon_1)} L_1$ is an isomorphism. Then there is a semi-orthogonal decomp

$$\mathcal{T} = \langle \operatorname{Im} L_2 \,,\ \operatorname{Im} C_2 \circ L_1 \,,\ \operatorname{Im} C_2 \circ C_1 \rangle$$

---

**Result:** $\Gamma_+ \to \mathbb{1} \to j_+$ $\qquad$ $\Phi_{\hat{Q}} \to \mathbb{1} \to \Phi_{\operatorname{Cone}(\hat{q})}$

"$C_1$" "$L_1$" "$C_2$" "$L_2$"

$$C_2 \circ C_1 = \Phi_{\hat{Q}} \circ \Gamma_+ = \Phi_{\hat{Q}_+} : \mathcal{Z}^{ss}_{S} \longrightarrow \mathcal{Z} \qquad \hat{Q}_+ := (j_+ \times \mathbb{1})^* \hat{Q}$$

Thus by above Lemma:

$L_1 C_1 \to L_1$ iso follows from

$\operatorname{Im}(\Phi_{\hat{Q}_+})$ is a full triangulated-sub category

Now add this to the fact that $\Phi_{\hat{Q}_+}$ is $\boxed{\text{faithful}}$ we have

This functor is faithful

it has a left inverse $j^*: D^b_{GL}(\mathrm{QCoh}(X)) \to D^b_{GL}(\mathrm{QCoh}(X^{ss}_+))$, to see this note that $X^{ss}_+$ has an affine chart consisting of the localization of the maximal minors of $B$, denote one of those by $t_I$ then w/ $I \subseteq \{1,\dots,m\}$, $|I|=n$ indexing the rows and note $R_{t_I} \otimes_S Q \cong K[GL] \otimes R$ as if you invert a minor of "$B_2$" from above you invert the determinant of $C$, and hence as $K[GL] \otimes R$ is the kernel to the identity and $j^* \circ \Phi_{\tilde{Q}^+}$

**Boom:** $\operatorname{Im}(\Phi_{\hat{Q}_+})$ is a window for $\mathcal{Z}_+$

**spoiler**: $\mathrm{Im}(\Phi_{\hat{Q}_+}) = \mathcal{K}$ full triangulated category generated by Kapranov's collection [The window from Don & Seg]

the important bit...

## Resolutions of $\hat{Q}_+$

(Not a foriegn concept:)

**Kernels connection to generation**

resolve the diagonal: $\mathcal{G}_\bullet \to \Delta \to 0$

$\mathcal{G}_\bullet$ is constructed by elements of the form $F_i \boxtimes G_i := \pi_X^* F_i \otimes \pi_Y^* G_i$

and when $\Delta \subseteq Y \times Y$ and $Y \to \mathrm{spec}(\mathbb{C})$ flat $\Rightarrow$ $\mathrm{Im}(\Phi_\Delta) \subseteq \langle G_i \rangle \subseteq D(Y)$

$\overset{\shortparallel}{\mathrm{Im}(\mathbb{1})}$

* By base change formula *

$\Phi_{F_i \boxtimes G_i}(M) = t_* t^*(F_i M) \boxtimes G_i$ *

when $\Delta \in D(\mathbb{P}^n \times \mathbb{P}^n)$ gives Beilinson's except. collection

when $\Delta \in D(Gr(n, \mathbb{C}^{n+k}) \times Gr(n, \mathbb{C}^{n+k}))$ gives Kapranov's except. collection

**Kempf's geometric method**:

**Lem**: $\exists$ a resolution of $\hat{Q}_+$ as a $Z_S^{ss} \times Z$-sheaf o' Modules such that each component is cone of elements from Kapranov's collection.

(terrible sketch in two pages)

**Cor**: $\mathrm{Im}(\Phi_{\hat{Q}_+}) \subseteq \mathcal{K}$

$M$

---

**Lem**: $\forall M \in \mathcal{K}$ $\Phi_{\hat{Q}_+}(i^* M) = M$

we show on vector bundles (since GL linearly reductive becomes as sum of irreps) that $\Phi_{\hat{Q}_+}$-acts like a "truncation" operator only leaving the polynomial-reps!

**Cor**: $\mathrm{Im}(\Phi_{\hat{Q}_+}) \supseteq \mathcal{K} \Rightarrow \mathrm{Im}(\Phi_{\hat{Q}_+}) = \mathcal{K}$

## Conclusion:

So what should we look for?

- Algebraically?
  - * Then what aspect?
    - how the images of $\sigma^{\#}$ and $p^{\#}$ cut out a variety?
    - the duality that continues to appear?
- partial compactifications?
  - perhaps searching for spaces with the semi-stable loci as boundaries?
- Drinfield?
  - do the closed points really control it all?
- windows?
  - then should we look for Bousfield (co) localizations?
  - Should we try and "build backwards" from resolutions?
    (perhaps with vanishing of sections?)

**Kempf's geometric method:** denote $\mathcal{S}$ as the tautlogical bundle over $Gr(d_1, W)$

and denote:

$$\begin{tikzcd} & Gr(d_1,W) \times Hom(W,V) \arrow[dl, "q"'] \arrow[dr, "f"] & \\ Gr(d_1,W) & & Hom(W,V) \end{tikzcd}$$

$$\begin{tikzcd} & 0 \arrow[r] & S \arrow[r] & W \arrow[r] & K \arrow[r] \arrow[d, squiggly] & 0 \\ & 0 \arrow[r] & q^*S \arrow[r] & q^*W \arrow[r] & q^*K \arrow[r] \arrow[d, squiggly] & 0 \\ 0 \arrow[r] & Hom(f^*V, q^*S) \arrow[r] & Hom(f^*V, q^*W) \arrow[r] & Hom(f^*V, q^*K) \arrow[r] \arrow[d, equal] & 0 & \\ & & & \Lambda & & \end{tikzcd}$$

Note

$$\mathcal{O}_{\underset{Gr \times Hom(W,V)}{Tot(Hom(f^*V, q^*S))}} \cong \hat{Q}_+$$

thus follows

$$\bigwedge^{rnk(\Lambda)} \Lambda \to \cdots \to \mathcal{O}_{Gr \times Z} \xrightarrow{q.is.} \hat{Q}_+$$ (as a resolution of $Gr \times Z$-modules)

and remains such when you lift to $Z^{ss}_{r} \times Z$-modules but important part is $\Lambda$ is a v.b. over $Z^{ss}_{j} \times Z$ so we use Schur's lemma to get our desired result.

**Vanishing of a Section:** The idea comes from:

$$Q \cong \mathbb{C}[A^L, A^R, B^L, B^R, C] \,/\, \langle B^L - B^R C,\ A^R - CA^L \rangle$$