Comparing Fractions etc.

The comparison test is a strong tool for determining the convergence or divergence of series using your knowledge of the convergence and divergence of 'easier' series.

Today we will look at some tools to make comparisons. The first thing we will look at is some helpful ideas used in many comparisons.

Helpful Comparisons

- *BIGGER bottom makes smaller*

$$\text{if} \quad a < b \quad \text{then} \quad \frac{1}{a} > \frac{1}{b}$$

- Consider a positive power $q > 0$. The is (some *BIG* number) $N > 0$ so that if $x \geq N$ then

$$\ln(x) \leq x^q \leq e^x$$

- The Trig bounds

$$-1 \leq \cos(x) \leq 1 \qquad \text{and} \qquad -1 \leq \sin(x) \leq 1$$

To use the comparison test it is helpful to know what we would like to compare to.

The big ones:

- *Geometric*: r^n — $\sum r^n$ is convergent ONLY when $|r| < 1$
- *p-series*: $\frac{1}{n^p}$ — $\sum \frac{1}{n^p}$ converges ONLY when $p > 1$

It is these two (the left column) that we will try and always find a comparison to.

OBJECTIVE!

Consider the sequence $\{a_n\}$ we want to find one of the following comparisons:

- Find an $|r| < 1$ so that $r^n < a_n$

- Find an $|r| > 1$ so that $a_n < r^n$

- Find a $0 < p < 1$ so that $\frac{1}{n^p} < a_n$

- Find a $p > 1$ so that $a_n < \frac{1}{n^p}$

Examples

- $$\frac{1}{3^n + n} < \frac{1}{3^n}$$

Solution: Note that $3^n < 3^n + n$ hence by the bigger bottom from above we have our answer.

- $$\frac{e^{-n}}{n + \cos^2(n)} \leq e^{-n}$$

Solution: From the trig bounds above we see that $\cos^2(n) \leq 1$ and hence $n + \cos^2(n) > n$ so by the bigger bottom property we have

$$\frac{e^{-n}}{n + \cos^2(n)} \leq \frac{e^{-n}}{n}$$

Further since $n \geq 1$ again by the bigger bottom property we thus now have our desired result.

- $$\frac{n}{n^2 - \cos^2(n)} > \frac{n}{n^2} = \frac{1}{n}$$

Solution: From the trig bounds above we see that $\cos^2(n) < 1$ and hence $n^2 - \cos^(n) < n^2$ so by the bigger bottom property we have our desired result.

Problem 1. Use the objectives in the previous box to find the appropriate comparison for the following.

i. $\dfrac{3^n + n}{2^{n+1}}$

ii. $\dfrac{4n-3}{2n^5}$

iii. $\dfrac{1}{(2n-1)(n-3)}$

iv. $\dfrac{\ln\left(n^2\right)}{n}$

v. $\dfrac{4n}{(n+1)^3}$

vi. $\dfrac{n-4}{(n^2+1)\,\mathbf{e}^n}$

vii. $\dfrac{\sqrt{2+\cos^2(5n)}}{\sqrt{n^2-n-1}}$

viii. $\dfrac{n-1}{\sqrt{n^3+n+3}}$

ix. $\dfrac{3n^2+7n-1}{n^4-n+3}$

x. $\dfrac{(1-\sin(n))\,(1+\sin(n))}{n^2+8n+1}$