Motivation:

Hard Question: when are two varieties derived equivalent?

Given two varieties X & Y, is $D^b(X) \cong D^b(Y)$?

* where to start looking? *

Conjecture: [Bondal & Orlov, Kawamata] (also hard)

Two complex projective varieties related by a flop are derived equivalent.

Audience Motivation:
- small contractions
- rational / birational morph.

my motivation

- [Ballard, Diemer, Favero] Suggest a kernel for this conjectured equivalence

what did they do?

* First They shifted the problem to V.G.I.T. *

- A D-flip [Reid] [a flop is a k-flip] is equivalent to a wall crossing of a $\mathbb{C}^*$-action

1) **Built a Functor:** ($\Bbbk \to$ a field)

$$\left[\begin{matrix}\text{category of varieties}\\ \text{w/ } \Bbbk^\times\text{-action}\end{matrix}\right] \quad \left[\begin{matrix}\text{category of varieties}\\ \text{w/ } (\Bbbk^\times)^2\text{-action}\end{matrix}\right]$$

$$\mathrm{Var}^{\Bbbk}_{\Bbbk^\times} \xrightarrow{\quad Q \quad} \mathrm{Var}^{\Bbbk}_{(\Bbbk^\times)^2}$$

partial compactification of $\Bbbk^\times$-action [uses a Monodial compactification of $\Bbbk^\times$]

- Technically we need to derive this functor in the sense of Derived alg. geo.

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partial Compactification:

$$\begin{tikzcd} & \tilde{Z} \arrow[d, shift right, "s"'] \arrow[d, shift left, "p"] \\ Z \times_* \Bbbk^\times \arrow[ur, "\iota"] \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\pi"'] & Z \end{tikzcd}$$

$Z \in \mathrm{Var}_{\Bbbk^\times}$, $* = \mathrm{Spec}(\Bbbk)$, $\sigma: Z \times \Bbbk^\times \to Z$ (action)

How? $\Bbbk^\times \overset{\mu}{\hookrightarrow} \mathbb{A}^1_{\Bbbk}$ (Monodial compactification)

ex: let $Z = \mathrm{Spec}(R)$

$$Q(Z) = \langle \sigma^\#(R),\ R \otimes_{\Bbbk} \Gamma(\mathbb{A}^1_{\Bbbk}, \mathcal{O}_{\mathbb{A}^1_{\Bbbk}}) \rangle \subseteq \Gamma(Z \times \Bbbk^\times, \mathcal{O}_{Z \times \Bbbk^\times})$$

Boundaries:

$$\partial := \tilde{Z} \setminus \iota(Z \times \Bbbk^\times)$$

un-stable & semi-stable loci:

$Z^{us}_p := p(\partial)$ $\qquad$ $Z^{us}_s := s(\partial)$

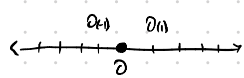
$Z^{ss}_p := Z \setminus Z^{us}_p$ $\qquad$ $Z^{ss}_s := Z \setminus Z^{us}_s$

V.G.I.T.: $\mathrm{Pic}_{G_m}(Z)$ $\quad G_m$-equivariant picard group

$\underline{\mathrm{Pic}}_{G_m}(Z) := \mathrm{Pic}_{G_m}(Z)/\sim \quad \mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}_{G_m}(Z)$ $\mathcal{L}_1 \sim \mathcal{L}_2$ iff $Z^{ss}(\mathcal{L}_1) = Z^{ss}(\mathcal{L}_2)$

ex: $Z = \mathrm{Spec}(R)$

$\mathcal{O}(m) \sim \mathcal{O}(n)$ iff m & n have same sign

Z: 

* $Z^{ss}(\mathcal{O}(1)) = Z^{ss}_s$ & $Z^{ss}(\mathcal{O}(-1)) = Z^{ss}_p$

2) push-foward:

$$\hat{Q}(Z) := (p \times s)_* \mathcal{O}_{\tilde{Z}} \in D^b_{(\mathbb{k}^*)^2}(Z \times Z)$$

3) Integral transform:

$$\Phi_{\hat{Q}(Z)} : D^b([Z/\mathbb{G}_m]) \longrightarrow D^b([Z/\mathbb{G}_m])$$

4) Semi-orthoganal Decomp:

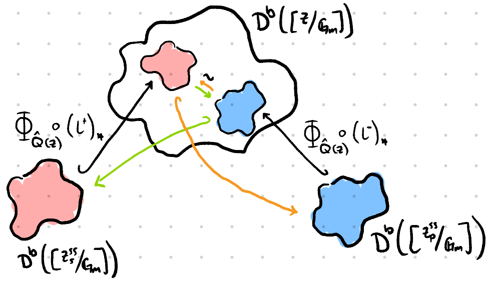
$$\iota^+ : Z^{ss}_s \hookrightarrow Z \qquad \iota^- : Z^{ss}_p \hookrightarrow Z$$

$$D^b([Z/\mathbb{G}_m]) =$$

$$\left\langle \left(\mathrm{Im}\, \Phi_{\hat{Q}} \circ (\iota^+)_*\right)^{\perp},\ \mathrm{Im}\, \Phi_{\hat{Q}} \circ (\iota^+)_* \right\rangle$$

$$\left\langle \left(\mathrm{Im}\, \Phi_{\hat{Q}} \circ (\iota^-)_*\right)^{\perp},\ \mathrm{Im}\, \Phi_{\hat{Q}} \circ (\iota^-)_* \right\rangle$$

5) Windows:



The Well-Crossing Kernel Program

Let G be a linear algebraic group over a field $\mathbb{k}$ such that $G \otimes_{\mathbb{k}} \overline{\mathbb{k}}_{sep}$ is linearly reductive

let $Z \in \mathrm{Var}^{\mathbb{k}}_G$ (varieties over $\mathbb{k}$ w/ G-action)

- we define a collection of monodal compactifications of G parameterized by $\mathrm{Pic}_G(Z)$

$$\mathcal{M} := \left\{ M_{[\mathcal{L}]} \,\middle|\, [\mathcal{L}] \in \mathrm{Pic}_G(Z) \right\} \qquad G \xhookrightarrow{\mu_{[\mathcal{L}]}} M_{[\mathcal{L}]}$$

- then define a collection of partial compactifications of the G-action on Z

$$\mathcal{C} := \left\{ \tilde{Z}_{[\mathcal{L}]} \,\middle|\, [\mathcal{L}] \in \mathrm{Pic}_G(Z) \right\}$$

$$\begin{tikzcd} & \tilde{Z}_{[\mathcal{L}]} \arrow[d, shift right, "s_{[\mathcal{L}]}"'] \arrow[d, shift left, "p_{[\mathcal{L}]}"] \\ Z \times G \arrow[ur, "\nu_{[\mathcal{L}]}"] \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "\pi"'] & Z \end{tikzcd}$$

- Boundaries:

$$\partial_{[\mathcal{L}]} := \tilde{Z}_{[\mathcal{L}]} \setminus \nu_{[\mathcal{L}]}(Z \times G)$$

- un-stable & semi-stable loci:

$$Z^{us}_{p_{[\mathcal{L}]}} := p_{[\mathcal{L}]}(\partial) \qquad Z^{us}_{s_{[\mathcal{L}]}} := s_{[\mathcal{L}]}(\partial)$$

$$Z^{ss}_{p_{[\mathcal{L}]}} := Z \setminus Z^{us}_{p_{[\mathcal{L}]}} \qquad Z^{ss}_{s_{[\mathcal{L}]}} := Z \setminus Z^{us}_{s_{[\mathcal{L}]}}$$

- wow:

$$Z^{ss}_{s_{[\mathcal{L}]}} = Z^{ss}(\mathcal{L})$$

$$Z^{ss}_{p_{[\mathcal{L}]}} = Z^{ss}(\mathcal{L}^{-1})$$

- **Push-forwards:**

$$Q_{[\lambda]} := \left(p_{[\lambda]} \times s_{[\lambda]}\right)_* \mathcal{O}_{\tilde{Z}_{[\lambda]}}$$

- **Integral Transform:**

$$\Phi_{Q_{[\lambda]}} : D^b([Z/G]) \longrightarrow D^b([Z/G])$$

- **Semi-orthogonal Decomp:**

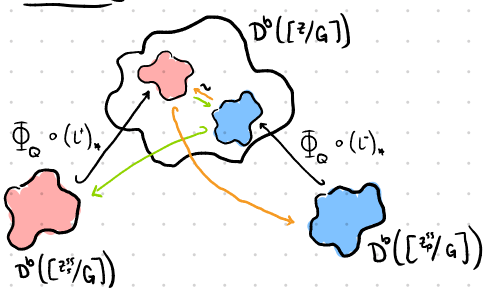
$$\iota^+_{[\lambda]} : Z^{ss}_{S_{[\lambda]}} \hookrightarrow Z \qquad \iota^-_{[\lambda]} : Z^{ss}_{P_{[\lambda]}} \hookrightarrow Z$$

$$D^b([Z/G]) =$$

$$\left\langle \left(\mathrm{Im}\, \Phi_{Q_\lambda} \circ (\iota^+_\lambda)_*\right)^\perp, \mathrm{Im}\, \Phi_{Q_\lambda} \circ (\iota^+_\lambda)_* \right\rangle$$

$$\left\langle \left(\mathrm{Im}\, \Phi_{Q_\lambda} \circ (\iota^-_\lambda)_*\right)^\perp, \mathrm{Im}\, \Phi_{Q_\lambda} \circ (\iota^-_\lambda)_* \right\rangle$$

- **Windows:**



no fixer needed!

- **But More:** Let $[\mathfrak{f}]$ and $[\mathfrak{g}]$ be two classes representing adjacent GIT chambers, w/ shared wall $\mathfrak{L}$

Then

$$(\iota_{\mathfrak{f}})^* \circ \Phi_{Q_{\mathfrak{g}}} \circ (\iota_{\mathfrak{g}})_* \quad \text{is an equivalence}$$

(under appropriate assumptions)

- **And More:**

might be leftovers...

$$D^b([Z/G]) \cong \left\langle (\text{others})^\perp, W_{[\lambda_i]} \;\middle|\; [\lambda_i] \text{ is a wall} \right\rangle$$

- The windows are defined by restrictions from chambers!

- **Still More:**

$$\Phi_{Q_{\mathfrak{f}}} \circ \Phi_{Q_{\mathfrak{g}}} = \Phi_{Q_{\mathfrak{f} \cup \mathfrak{g}}}$$

in particular $\Phi_{Q_{\mathfrak{f}}} \circ \Phi_{Q_{\mathfrak{f}}} = \Phi_{Q_{\mathfrak{f}}}$

Can define for a larger class
- might not be Monoids -

Available (on Arxiv) showing examples:

- [Ballard, Diemer, Favero] w/ the group $\mathbb{G}_m$

They only show one compactification, but the other non-trivial one only 2 since only two chambers

$$Q^-(R) = \langle \sigma^*(R), \pi^*(R), u^{-1} \rangle$$

- [Ballard, Chidambaram, Favero, McFaddin, V–] w/ the group GL_n (Grassmann Flops)

again we only show one, there are **many** still working those out!

Example(s):

- [Ballard, V-] w/ the group $\mathbb{G}_m^n$ (Let's see that!)
  [still Hammering out some details]

- Let V be a d-dimensional $\Bbbk$-vector space w/ $\mathbb{G}_m^n$-action
- Denote $Z := \operatorname{Spec}(V^\vee)$ inherits $\mathbb{G}_m^n$-action
- Equivalent to giving $\operatorname{Sym}(V^\vee)$ a $\mathbb{Z}^n$-grading
- $\underline{\operatorname{Pic}}_{\mathbb{G}_m^n}(Z) \cong \mathbb{Z}^n$

Specific Example $\quad n=2, \ d=4$

- $V = \operatorname{Span}_{\Bbbk}\{e_1, \dots, e_4\}$
- $R := \Bbbk[x_1, \dots, x_4] = \operatorname{Sym}(V^\vee)$
- $Z = \operatorname{Spec}(\Bbbk[x_1, \dots, x_4])$
- $\underline{\operatorname{Pic}}_{\mathbb{G}_m^2}(Z) \cong \mathbb{Z}^2$

The action: $\sigma : Z \times \mathbb{G}_m^n \longrightarrow Z$

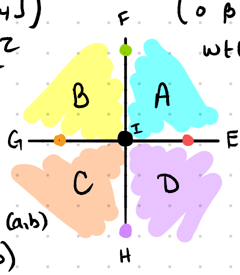
$\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix} e_1 := \alpha e_1 \qquad \begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix} e_2 := \beta e_2$

$wt(x_3) = (-1, 0) \qquad wt(x_4) = (0, -1)$

$\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix} e_3 := \alpha^{-1} e_3 \qquad \begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix} e_4 := \beta^{-1} e_4$

$wt(x_1) = (1, 0) \qquad wt(x_2) = (0, 1)$

$\underline{\operatorname{Pic}}_{\mathbb{G}_m^2}(Z) \cong$



i.e. each point (a,b) "represents" $\mathcal{O}(a,b)$

[$\mathcal{L} \sim \mathcal{L}'$ iff in same shaded Region]

- Monodal Compactifications: $\mathcal{J} = \{A, B, C, D, E, F, G, H, I\}$

$$\mathcal{M} = \{ M_j \mid j \in \mathcal{J} \} \qquad M_j \subseteq \Bbbk[\mathbb{G}_m] = \Bbbk[t_1^{\pm}, t_2^{\pm}] \quad \forall j \in \mathcal{J}$$

full dimensional

- $M_A = \operatorname{Spec}(\Bbbk[t_1^{-1}, t_2^{-1}])$
- $M_C = \operatorname{Spec}(\Bbbk[t_1, t_2])$
- $M_B = \operatorname{Spec}(\Bbbk[t_1, t_2^{-1}])$
- $M_D = \operatorname{Spec}(\Bbbk[t_1^{-1}, t_2])$

$M_E = \operatorname{Spec}(\Bbbk[t_1^{-1}, t_2^{\pm}])$

$M_F = \operatorname{Spec}(\Bbbk[t_1^{\pm}, t_2^{-1}])$

$M_G = \operatorname{Spec}(\Bbbk[t_1, t_2^{\pm}]) \qquad M_H = \operatorname{Spec}(\Bbbk[t_1^{\pm}, t_2]) \qquad M_I = \mathbb{G}_m^n$

- partial compactifications:

$$\mathcal{C} = \{ \tilde{Z}_j := \operatorname{Spec}(R_j) \mid j \in \mathcal{J} \} \qquad R_j = \langle \sigma^{\#}(\Bbbk[Z]), \Bbbk[Z], \Bbbk[M_j] \rangle \subseteq \Bbbk[Z \times \mathbb{G}_m^n]$$

- **Boundaries:** $\forall j \in \mathcal{J}$ $\quad \iota_j : Z \times \mathbb{G}_m^n \longrightarrow \tilde{Z}_j$ is induced by the inclusion

(for full dim) $$\partial_j = \operatorname{Spec}\left(R_j / \langle \text{gen of } M_j \rangle\right)$$

Lower dimensionals

$$\partial_E = \operatorname{Spec}\left(R_j / \langle t_1^{-1} \rangle\right)$$

$$\partial_F = \operatorname{Spec}\left(R_j / \langle t_2^{-1} \rangle\right)$$

$$\partial_G = \operatorname{Spec}\left(R_j / \langle t_1 \rangle\right)$$

$$\partial_H = \operatorname{Spec}\left(R_j / \langle t_2 \rangle\right)$$

$$\partial_I = \operatorname{Spec}(R_j)$$

- **unstable Loci:**

$$Z^{us}_{S_j} = \operatorname{Spec}\left(R_j / \langle r \mid s^{\#}(r) \cap \text{defining ideal o' } \partial_j \neq \emptyset \rangle\right)$$

$$Z^{us}_{P_j} = \operatorname{Spec}\left(R_j / \langle r \mid p^{\#}(r) \cap \text{defining ideal o' } \partial_j \neq \emptyset \rangle\right)$$

- **Semi-Stable Loci:**

$$Z^{ss}_{S_j} = \bigcup_{\substack{x_i \in \text{defining} \\ \text{ideal of } Z^{ss}_{S_j}}} \operatorname{Spec}(R_{x_i})$$

$$Z^{ss}_{P_j} = \bigcup_{\substack{x_i \in \text{defining} \\ \text{ideal of } Z^{ss}_{S_j}}} \operatorname{Spec}(R_{x_i})$$

- **The Cool Stuff:**

$$\left.\begin{aligned} Z^{ss}(\ell) &= Z^{ss}_{P_j} \\ Z^{ss}(-\ell) &= Z^{ss}_{S_j} \end{aligned}\right\} \text{ iff } \ell \in j$$

- **windows**

$$D^b\left(\left[Z / \mathbb{G}_m^n\right]\right) = \left\langle \mathcal{W}_j \;\middle|\; j \in \mathcal{J} \text{ is 1-dim} \right\rangle$$

where $$\mathcal{W}_j = \bigcap_{i \cap j \neq \emptyset} \operatorname{Im}\left(\Phi_{Q_i}\right)$$

Future Directions:

- Big Question what happens when Z is singular???

  - $GL_n \times \mathbb{G}_m$
  - $Sp_{2n} \times \mathbb{G}_m$

  } GLSM w/ the help of Matrix factorizations...