Poly....

$3x^2+3x+2x^7+8x^0$

$ax^n+bx^{n-1}+\dots$

**Chapter 3**
**Section 3.5**
**Warm-up Problem A.** Write each of the following as a single fraction:

(a) $\frac{u}{u^4}+\frac{1}{u^2}\cdot\frac{u^2}{u^2}$

$\frac{u}{u^4}+\frac{u^2}{u^4}=\frac{u+u^2}{u^4}=\frac{u(1+u)}{u^4}$

$=\frac{(1+u)}{u^3}$

(b) $\frac{x}{\boxed{(x+3)}\boxed{(x-3)}}+\frac{2}{x-3}\cdot\frac{(x+3)}{(x+3)}$

$=\frac{x+2(x+3)}{(x+3)(x-3)}=\frac{x+2x+6}{(x+3)(x-3)}=\frac{3x+6}{(x+3)(x-3)}$

$=\frac{3(x+2)}{(x+3)(x-3)}$

**Main Topic # 1:** [Rational Functions]

A Rational Function is a polynomial function divided by a non-zero polynomial function.

**Learning Outcome # 1:** [Be able to tell the difference between a rational function and not a rational function]

**Problem 1.** Determine whether the following functions are rational functions. If so, write them in the form $p(x)/q(x)$, where $p$ and $q$ are polynomials.

a) $f(x)=\frac{2x^3+7}{x^0}$ ← poly ← div ← poly

c) $f(x)=\frac{x^2}{2-x}-\frac{1}{x-2}$

$-1(x-2)$

$\frac{-x^2}{x-2}-\frac{1}{x-2}=\frac{-x^2-1}{x-2}$

$x^2$

$2^x$

b) $g(x)=\frac{2^x+1}{2^x-2}$

d) $g(x)=\frac{\sqrt{x}+3}{\sqrt{x}}$ → $x^{1/2}+3$

$x^{-1}=\frac{1}{x}$

**Main Topic # 2:** [Holes]

A rational function is the one type of function which has a restricted domain. One way to have a restricted domain which will appear in the graph as a Hole. A hole happens when you can "cancel out" the "bad point(s)"

**Hole Example**

Consider the following rational function:

$$F(x)=\frac{x^2+6x+9}{x+3}$$

Since division by zero is not well defined (mathy lingo for doesn't make sense) we see that we can't plug in -3, that is the domain is all real numbers $x$ such that $x\neq-3$. Yet notice we can "cancel out" the $x+3$ from the bottom (denominator)

$$\frac{x^2+6x+9}{x+3}=\frac{(x+3)\cancel{(x+3)}}{\cancel{(x+3)}}=x+3$$

$x\neq-3$

The graph of a rational function with a hole looks like the "canceled out" graph, that is the graph of the function associated to the equation obtained after canceling out.

## Hole Graphing Example

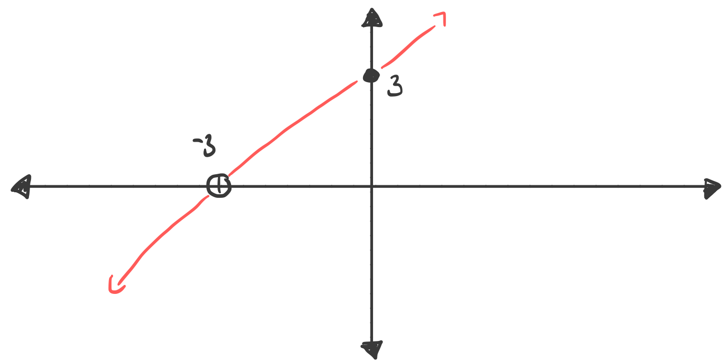
Consider the rational function from the previous example:

$$F(x) = \frac{x^2 + 6x + 9}{x + 3}$$

Recall this has a hole at $x = -3$. Once we did the canceling we saw for any value of $x$ so that $x \neq -3$ we had:

$$\frac{x^2 + 6x + 9}{x + 3} = x + 3$$

Which is just a line but we cannot plug in -3 so it's graph looks like: Notice the literal <u>hole</u>




**Main Topic # 3:** [Vertical Asymptote]

Another way a rational function may have a restricted domain will appear in the graph as a <u>Vertical Asymptote</u>. A vertical asymptote happens where a value of $x$ makes the denominator zero (but cannot cancel out)

## Vertical Asymptotes

The graph of a rational function will have a vertical asymptote at $x = a$ if the denominator is zero at $x = a$ but the numerator is **not** zero at $x = a$

The part about the numerator not being zero assures there is no canceling out.

## Vertical Asymptote Example

Consider the following rational function:

$$f(x) = \frac{2x^2 + x + 3}{3x^2 - 5x - 7}$$

This function has zeros of the denominator i.e. $0 = 3x^2 - 5x - 7$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{25 - 4(3)(-7)}}{6} = \frac{5 \pm \sqrt{109}}{6}$$

$$x = \frac{5 + \sqrt{109}}{6} \text{ or } x = \frac{5 - \sqrt{109}}{6}$$

And this function has no "Real" zeros in the numerator i.e. $0 \neq 2x^2 + x + 3$ since:

$b^2 - 4ac$ ← discriminant $\quad 1 - 4(2)(3) = -23$

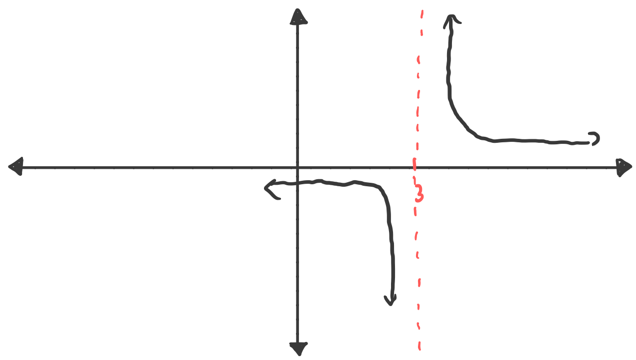
So it has a vertical asymptote(s) at $x = \frac{5 + \sqrt{109}}{6}, \frac{5 - \sqrt{109}}{6}$

### Graphing Vertical Asymptotes

When graphing a vertical asymptote at $x = a$ it is like the graph is driving towards a wall at the line vertical line $x = a$ and must veer to one side... Consider

$$f(x) = \frac{1}{x-3}$$

It has a vertical asymptote at $x = 3$ since 3 makes the denominator zero but not the numerator so the graph looks like: (we will see how to sketch these graphs further later)




**Main Topic # 4:** [End Behavior]

The End Behavior for rational functions is tricky... they can sometimes be described with horizontal asymptotes, when they cannot be described by these they go off to infinity just like polynomials.

### Horizontal Asymptotes

For a rational function: ($a$ and $b$ are non-zero real numbers)

Degree of numerator → $n$ (Top)

$$f(x) = \frac{ax^n + \dots}{bx^{m} + \dots}$$

Look Typo!

Degree of denominator → $m$ (Bottom)

Smaller Top
1. If $n < m$ then the $x$-axis ($y = 0$) is the horizontal asymptote.

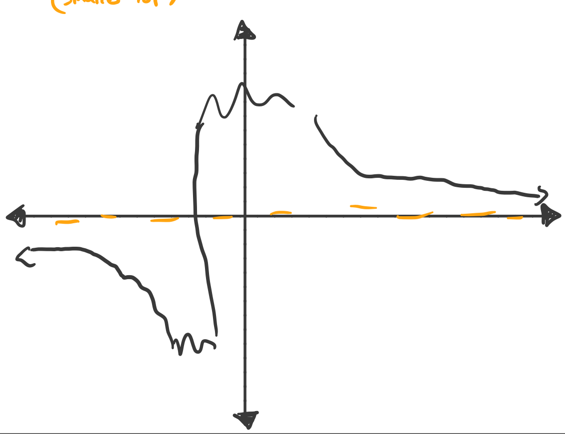
equal Top & Bottom
2. If $n = m$ then the line $y = \frac{a}{b}$ is the horizontal asymptote.

Bigger Top
3. If $n > m$ there are no horizontal asymptotes.

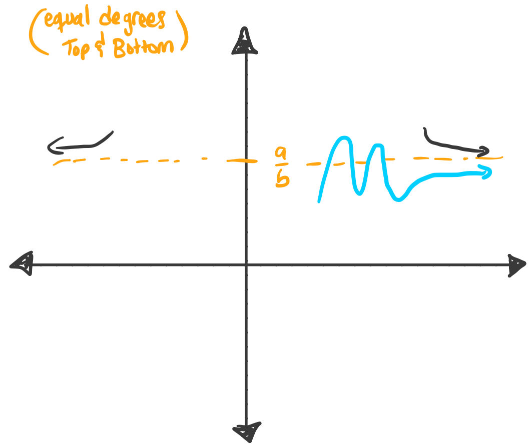
### Graphing Horizontal Asymptotes

The Possible ways this can look on a graph are the following:

$n < m$
(Smaller Top)



$n = m$
(equal degrees Top & Bottom)



**Learning Outcome # 2:** [Be able to find asymptotes, holes, and end behavior of rational functions.]

**Problem 2.** Compare and discuss the end behaviors of the following functions. Identify any holes or horizontal and vertical asymptotes.

(a) $f(x) = \dfrac{2x^2+1}{x^2+2}$

$b^2-4ac$

$0-4(1)(2)$

$=-8$

(b) $g(x) = \dfrac{4x^3+2x}{2x^2+x+1}$

No: VA or holes

No: HA

(c) $h(x) = \dfrac{x^2-5x+6}{(x-2)(x+5)}$

hole: $x=2$

VA: $x=-5$

HA: $y=1$

**Problem 3.** Evaluate the following limits of rational functions: (Recall this means "what's the end behaviors?")

(a) $\lim_{x\to\infty}(10+x^{-2}) = 10$

$10+\frac{1}{x^2}$

$\frac{10x^2+1}{x^2}$

(b) $\lim_{x\to-\infty}\dfrac{3x^4+10x^5}{2x^7+4} = 0$

(c) $\lim_{x\to\infty}\left(\dfrac{5x^3}{x^3+2x^2}+1\right) = \frac{6}{1} = 6$

$\dfrac{5x^3+x^3+2x^2}{x^3+2x^2}$

$=\dfrac{6x^3+2x^2}{x^3+2x^2}$

**Main Topic # 5:** [Sketching Graphs of Rational Functions]

**Sketching Rational Functions**

1. Find the intercepts, if there are any. Remember that the $y$-intercept is given by $(0, f(0))$ i.e. plugging in zero for $x$, and we find the $x$-intercepts by **setting the numerator equal to zero** and solving.
2. Find the vertical asymptotes and holes by setting the denominator equal to zero and solving.
3. Find the horizontal asymptote, if it exits, using the facts above.
4. The vertical asymptotes will divide the number line into regions. In each region graph at least one point in each region. This point will tell us whether the graph will be above or below the horizontal asymptote and if we need to we should get several points to determine the general shape of the graph.
5. Sketch the graph.

**Learning Outcome # 3:** [Sketching/Identifying graphs of rational functions.]

**Problem 4.** Determine the end behavior of the following rational functions. Then, determine their $x$-intercepts, $y$-intercepts, and horizontal/vertical asymptotes, if there are any. Then Sketch the graph.

(a) $y = \dfrac{x+3}{x+5}$

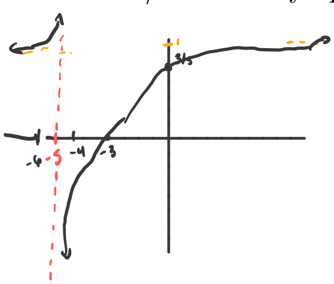
x-inter → $(-3,0)$

y-inter → $(0, \frac{3}{5})$

VA: $x=-5$

HA: $y = \frac{1}{1} = 1$

hole: none




(c) $h(x) = \dfrac{x^2-4}{x^3+4x^2} = \dfrac{(x-2)(x+2)}{x^2(x+4)}$

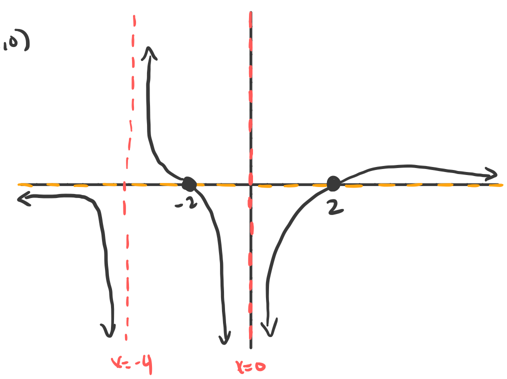
X-inte: $(2,0)$, $(-2,0)$

y-inte: None

HA: $y=0$

VA: $x=0,-4$

hole: none




(b) $y = \dfrac{x+2}{(x-1)^2}$

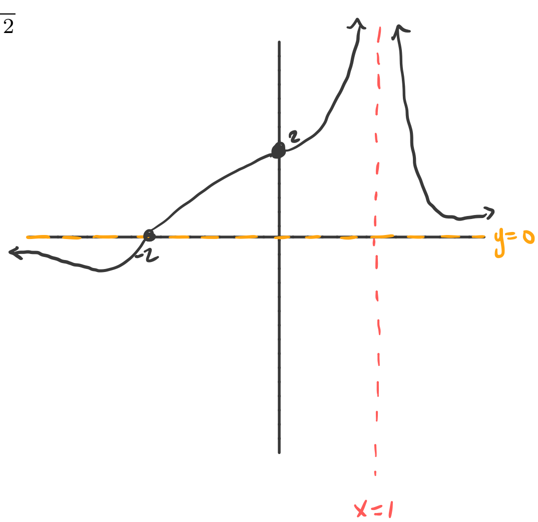
x-int: $(-2,0)$

y-int: $(0,2)$

HA: $y=0$

VA: $x=1$

hole: none




(d) $k(x) = \dfrac{x^2-9}{x^2-3x+2} = \dfrac{(x-3)(x+3)}{(x-2)(x-1)}$

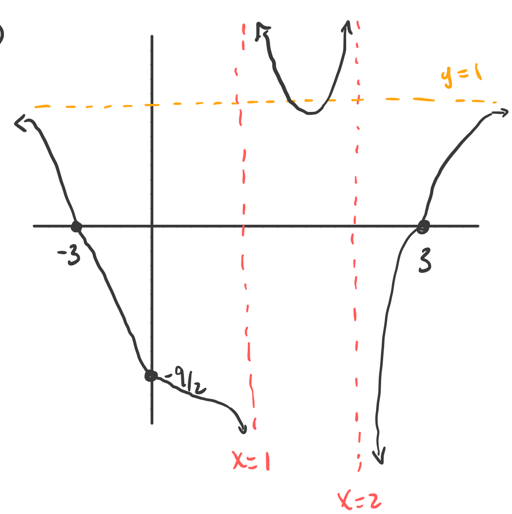
x-int: $(3,0)$, $(-3,0)$

y-int: $(0, \frac{-9}{2})$

HA: $y=1$

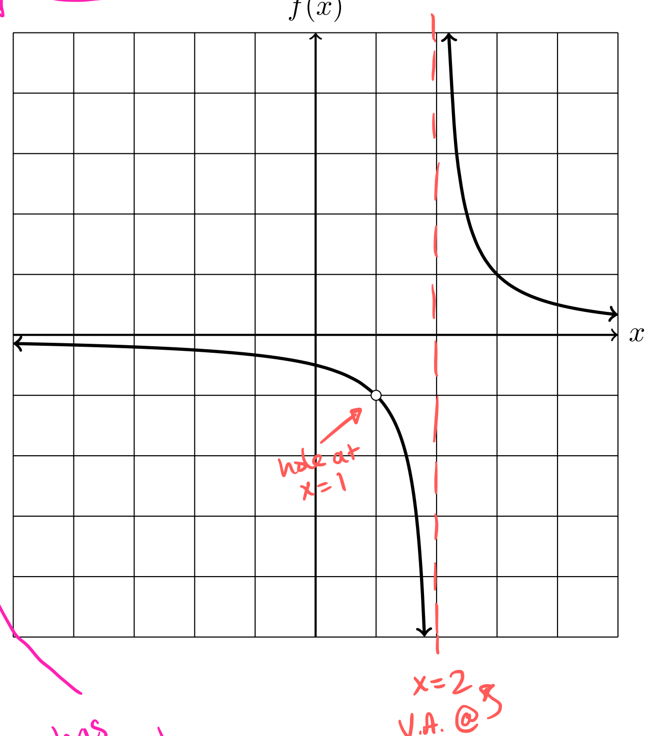
VA: $x=2,1$

hole: none



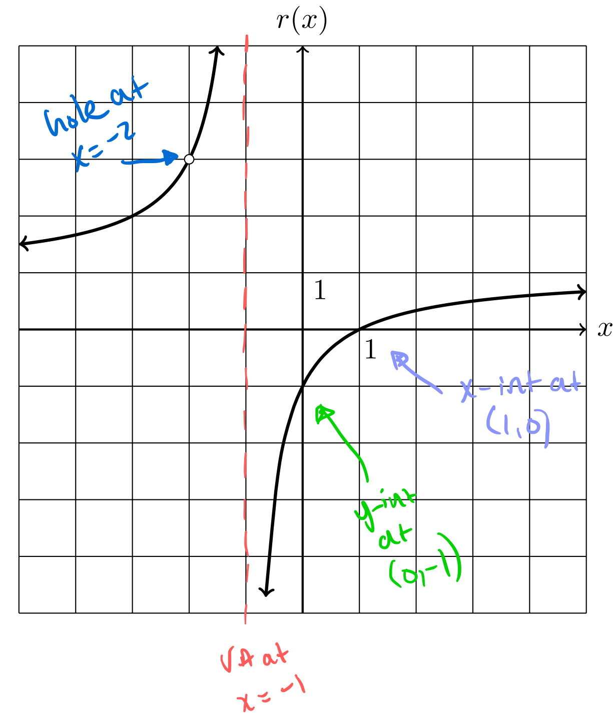
**Problem 5.** Determine which of the following functions matches the given graph.

$$y = \frac{x-1}{(x-2)(x-1)} \quad \text{or} \quad y = \frac{x-1}{(x+2)(x-1)}$$




has hole at x=1 and VA at x=2 !

**Problem 6.** Determine a possible formula for the rational function whose graph is shown below.




$$\frac{(x-1)(x+2)}{(x+1)(x+2)} = f(x)$$

The Blue make a hole at x=-2

The Red makes a V.A. at x=-1

The Purple makes x-int (1,0)

Notice when you plug in zero for x (i.e. y-int) you get

f(0) = -1

So has correct y-int!