Bases consisting of rational functions of uniformly bounded degrees or more general functions *

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Abstract

We prove in this paper the existence of a Schauder basis for $C[0,1]$ consisting of rational functions of uniformly bounded degrees. This solves an open question of some years concerning the possible existence of such bases. This result follows from a more general construction of bases on $\mathbb{R}$ and $[0,1]$. We prove that the new bases are unconditional bases for L_p, $1 < p < \infty$, and Besov spaces. On $[0,1]$, they are Schauder bases for $C[0,1]$ as well. The new bases are utilized to nonlinear approximation.

1 Introduction

One of our main goals in this paper is to solve the problem for existence of a Schauder basis for $C[0,1]$ consisting of rational functions of uniformly bounded degrees. This problem was posed in [S] and [DS]. Shekhtman [S] proved that there exists a Schauder basis $\{r_n\}_{n=0}^{\infty}$ for $C[0,1]$ so that r_n is a rational function with $\deg r_n = O(\ln^2 n)$. We shall prove that there exists a rational Schauder basis $\{r_n\}_{n=0}^{\infty}$ for $C[0,1]$ such that $\deg r_n \le K < \infty$. There is a striking difference between rational and polynomial bases for $C[0,1]$. If $\{p_n\}_{n=0}^{\infty}$ is a polynomial Schauder basis for $C[0,1]$ and $\deg p_n \le \deg p_{n+1}$, $n = 0, 1, \dots$, then $\deg p_n \approx n$ is the best possible, see [Pr1], [Pr2]. For orthogonal polynomial bases see [LS], [WW].

The techniques we develop in this paper can actually be applied in a quite general setting and to a variety of function spaces. Our technique will give a new method for constructing bases that are unconditional for L_p $(1 < p < \infty)$, Besov, and other spaces, and Schauder bases for $C[0,1]$.

Our idea for constructing bases stems from the well known idea of a small perturbation argument: Given a basis $\{\psi_j\}_{j=1}^{\infty}$ for some Banach space X, if the functions θ_j approximate ψ_j well enough, then $\{\theta_j\}_{j=1}^{\infty}$ will also be a basis for X. The key question is: In what sense should ψ_j be approximated by θ_j? If one elects to make $\|\psi_j - \theta_j\|_X$ small enough, then $\{\theta_j\}_{j=1}^{\infty}$ is automatically a basis for X (see, e.g. [LT]). However, there is not much room for maneuvering when selecting θ_j. Our approach is different. We start (on $\mathbb{R}$) from an excellent

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orthonormal wavelet basis $\{\psi(2^k t-j)\}$ with compactly supported ψ with enough smoothness and vanishing moments.Then we select θ smooth with enough vanishing moments so that

$$|\psi^{(j)}(t) - \theta^{(j)}(t)| \le \varepsilon(1 + |t|)^{-M} \quad \text{for} \quad t \in \mathbb{R}, \quad j = 0, 1, \dots, k,$$

where $\varepsilon > 0$ is small enough (fixed) and $M > 0$ and $k > 0$ are big enough. We prove that $\{\theta(2^k t-j)\}$ is an unconditional basis for L_p $(1 < p < \infty)$ and other spaces. Our construction of bases on $[0, 1]$ is similar. In this case, however, the basis functions are not dyadic shifts and dilates of a single function. We prove that our bases on $[0, 1]$ are Schauder bases for $C[0, 1]$ as well. The trade off is that we give up the orthogonality and multiresolution analysis but preserve all other good properties of the wavelets and gain much more flexibility in selecting the basis functions.

Our main application of this new small perturbation technique is to the construction of bases consisting of functions that are linear combinations of a fixed (small) number of shifts and dilates of a single function Φ. This function ought to be smooth enough and with sufficiently rapid decay. For instance, the rational function $\Phi(t) = (1 + t^2)^{-m}$ with m big enough generates the desirable rational bases. Another interesting example is the Gaussian $\Phi(t) = e^{-t^2}$.

Another important motivation for our work in constructing bases is nonlinear approximation. It has been well understood in approximation theory that unconditional bases for L_p $(1 < p < \infty)$, Besov, and other spaces provide a simple and powerful tool for nonlinear approximation. Namely, suppose that $\{\psi_j\}_{j=1}^{\infty}$ is such a basis. Then each function $f \in L_p$ can be represented by $f = \sum c_j \psi_j$. It is natural to consider approximation of f by linear combinations of n basis functions ψ_j (n-term approximation). The strategy for achieving best or near best n-term approximation to f is simply to retain the n terms from the expansion of f with the biggest $\|c_j \psi_j\|_{L_p}$. It turns out that (under mild conditions on $\{\psi_j\}$) the n-term approximation can be characterized by Besov and other spaces. The above leads us to the following idea for nonlinear approximation: Suppose that we want to approximate by linear combinations of functions from some approximating family $\mathbf{D}$. Then we can proceed as follows. First, we construct a good basis which elements are linear combinations of a fixed number of functions from $\mathbf{D}$ and, secondly, we run the best n-term approximation algorithm described above. We refer the reader to [De] as a general reference for nonlinear n-term approximation.

We decided not to consider bases for other spaces besides $C[0, 1]$, and the univariate L_p $(1 < p < \infty)$ and Besov spaces in this paper. We shall report our results about H^p and other spaces, and in the multivariate case elsewhere.

The outline of the paper is the following. In §2, we give the construction of the new bases (systems). In §3, we give the basic properties of the new systems. In §4, we prove one of the main result of the paper. Namely, the new system for $[0, 1]$ is a Schauder basis for $C[0, 1]$. In §5, we prove that the new systems are unconditional bases for L_p, $1 < p < \infty$, and Besov spaces. In §6, we prove the needed approximation result for the construction of bases consisting of linear combinations of shifts and dilates of a single function. In §7, we give examples of new bases and, in particular, rational bases. We utilize them to nonlinear n-term approximation. §8 is an appendix, where we give the proofs of some technical statements from §3 - §5.

Throughout the paper, the constants are denoted by $C, C_1, \dots$ and they may vary at every occurrence. The constants usually depend on some parameters that will be sometimes indicated explicitly.

2 Construction of new systems (bases)

We shall construct our bases (the new systems) by using as a backbone excellent wavelet bases (the old bases). We shall use as old bases smooth compactly supported orthogonal wavelet bases with enough vanishing moments (Daubechies wavelets) although other wavelet bases can be used as well. We shall have two variants of our construction, namely, on $\mathbb{R}$ and on the compact interval $[0,1]$.

We first introduce some notation. Let $\Omega = \mathbb{R}$ or $\Omega = [0,1]$. Let $\mathcal{D} := \mathcal{D}(\Omega)$ denote the collection of all dyadic subintervals of Ω and let

$$\mathcal{D}_m := \mathcal{D}_m(\Omega) := \{I \in \mathcal{D} : |I| = 2^{-m}\},$$

where $|I|$ denotes the length of I. Thus

$$\mathcal{D}(\mathbb{R}) = \bigcup_{m\in\mathbb{Z}} \mathcal{D}_m \quad \text{and} \quad \mathcal{D}([0,1]) = \bigcup_{m\geq 0} \mathcal{D}_m.$$

For each $I \in \mathcal{D}$, we let t_I denote the left end of the interval I.

• **An old basis on $\Omega = \mathbb{R}$.** Let N and k be positive integers so that $N \geq k+1$ and let $A > 1$. Let $\mathcal{A} := \mathcal{A}(\mathbb{R})$ be an orthonormal wavelet basis consisting of compactly supported smooth wavelets and constructed from a multiresolution analysis generated by a compactly supported scaling function ϕ (Daubechies compactly supported wavelets). More precisely, we assume that there exists a ladder of closed subspaces of $L_2(\mathbb{R})$

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$$

with

$$\overline{\bigcup_{m\in\mathbb{Z}} V_m} = L_2(\mathbb{R}) \quad \text{and} \quad \bigcap_{m\in\mathbb{Z}} V_m = \{0\}$$

so that

(a) $f \in V_m \Leftrightarrow f(2^m x) \in V_0$ and

(b) $\{\phi(t-\nu)\}_{\nu\in\mathbb{Z}}$ is an orthonormal basis for V_0.

Let ψ be the mother wavelet. That is $\{\psi(t-\nu)\}_{\nu\in\mathbb{Z}}$ is an orthonormal basis for $W_0 := V_1 \ominus V_0$.

We denote, for each $I \in \mathcal{D}$,

$$\phi_I(t) := |I|^{-1/2}\phi\left(\frac{t-t_I}{|I|}\right) \quad \text{and} \quad \psi_I(t) := |I|^{-1/2}\psi\left(\frac{t-t_I}{|I|}\right).$$

Then $\mathcal{A} = \{\psi_I\}_{I\in\mathcal{D}}$ and $\mathcal{A}$ is an orthonormal basis for $L_2(\mathbb{R})$, $\{\phi_I\}_{I\in\mathcal{D}_m}$ is an orthonormal basis for V_m, and $\{\psi_I\}_{I\in\mathcal{D}_m}$ is an orthonormal basis for $W_m := V_{m+1} \ominus V_m$.

In addition to this, let ϕ and ψ satisfy the following properties with $\Omega = \mathbb{R}$:

$$\phi, \psi \in C^N(\Omega), \tag{2.1}$$

$$\int_\Omega t^\nu \psi(t)\, dt = 0, \quad \nu = 0, 1, \ldots, k-1, \tag{2.2}$$

and

$$\text{Supp } \phi, \text{ Supp } \psi \subset [-A, A]. \tag{2.3}$$

Simple change of variables shows that (2.1) - (2.3) yield that ϕ_I and ψ_I satisfy the following properties: For $I \in \mathcal{D}(\Omega)$,

A1.

$$\phi_I, \psi_I \in C^N(\Omega)$$

and

$$\|\phi_I^{(j)}\|_{L_\infty(\Omega)}, \ \|\psi_I^{(j)}\|_{L_\infty(\Omega)} \le C_j |I|^{-j-1/2}, \quad j = 0, 1, \ldots, N;$$

A2.

$$\int_\Omega t^\nu \psi_I(t)\, dt = 0, \quad I \in \mathcal{D}(\Omega), \quad \nu = 0, 1, \ldots, k-1;$$

A3.

$$\text{Supp } \phi_I, \text{Supp } \psi_I \subset [t_I - A|I|, t_I + A|I|].$$

Also, we assume that the following condition holds:

A4. $\mathcal{A}$ is an unconditional basis for $L_p(\Omega)$, $1 < p < \infty$, and the Besov space $B^s_q(L_p(\Omega))$ that will be specified later in §5.

Daubechies wavelets of sufficiently high smoothness provide a basis like this, see [Da]. For the most parts of this paper, condition (2.3) can be relaxed. It can be replaced by

$$|\phi(t)|, |\psi(t)| \le \frac{C}{(1 + |t|)^S}, \quad t \in \mathbb{R},$$

with S large enough. Then Mayer's wavelets as well as smooth spline or other wavelet basses can be used as old bases (see [Da], [HW], [Me], [W]).

• **Construction of a new system (basis) on $\Omega = \mathbb{R}$.** Let N and k be the parameters of the old basis. Let $M > 1$ and $\varepsilon > 0$. We select a function $\theta \in C^k(\mathbb{R})$ that satisfies the following conditions:

$$|\psi^{(j)}(t) - \theta^{(j)}(t)| \le \varepsilon \left(1 + |t|\right)^{-M}, \quad t \in \mathbb{R}, \quad j = 0, 1, \ldots, k, \tag{2.4}$$

and

$$\int_\Omega t^\nu \theta(t)\, dt = 0, \quad j = 0, 1, \ldots, k-1. \tag{2.5}$$

We define

$$\theta_I(t) := |I|^{-1/2} \theta\left(\frac{t - t_I}{|I|}\right), \quad I \in \mathcal{D}.$$

Simple change of variables in (2.4) and (2.5) shows that θ_I satisfies the following properties:

B1.

$$|\psi_I^{(j)}(t) - \theta_I^{(j)}(t)| \le \varepsilon |I|^{-j-1/2} \left(1 + \frac{|t-t_I|}{|I|}\right)^{-M}, \quad t \in \mathbb{R}, \quad j = 0, 1, \dots, k,$$

and

B2.

$$\int_{\mathbb{R}} t^\nu \theta_I(t)\, dt = 0, \quad j = 0, 1, \dots, k-1.$$

By **A1**, **A3**, and **B1**, we obtain the following decay property

$$|\psi_I^{(j)}(t)|, |\theta_I^{(j)}(t)| \le C|I|^{-j-1/2} \left(1 + \frac{|t-t_I|}{|I|}\right)^{-M}, \quad t \in \mathbb{R}, \quad j = 0, 1, \dots, k. \tag{2.6}$$

We define now the new system $\mathcal{B}$ by

$$\mathcal{B} := \mathcal{B}(\mathbb{R}) := \{\theta_I\}_{I \in \mathcal{D}(\mathbb{R})}.$$

It will be shown in §6 that functions $\theta \in C^k(\mathbb{R})$ that satisfy (2.4) and (2.5) exist. Therefore, new systems exist.

• **An old basis on** $\Omega = [0, 1]$**.** Let again N and k be positive integers so that $N > k + 1$ and let $A > 1$. Let

$$\mathcal{A} := \mathcal{A}([0, 1]) := \{\phi_I\}_{I \in \mathcal{D}_{m_0}} \cup \{\psi_I\}_{I \in \cup_{m \ge m_0} \mathcal{D}_m}, \quad m_0 > 0,$$

be an orthonormal wavelet basis for $L_2[0, 1]$ with the following properties: there exists a ladder of finite dimensional subspaces of $L_2[0, 1]$

$$V_{m_0} \subset V_{m_0+1} \subset \cdots$$

with

$$\overline{\cup_{m \ge m_0} V_m} = L_2[0, 1]$$

so that

$$V_m = \text{Span}\ \{\phi_I\}_{I \in \mathcal{D}_m} \ \text{ and } \ W_m := V_{m+1} \ominus V_m = \text{Span}\ \{\psi_I\}_{I \in \mathcal{D}_m}, \quad m = m_0, m_0 + 1, \dots,$$

with ϕ_I and ψ_I satisfying properties **A1** - **A4** with $\Omega = [0, 1]$ (see the properties of the old basis on $\Omega = \mathbb{R}$). In addition to this we assume that $\mathcal{A}$ satisfies the following property:

A5. $\mathcal{A}$ is a Schauder basis for $C[0, 1]$.

When the basis functions of $\mathcal{A}$ need to be ordered, we assume that they are ordered from low to high levels and from left to right on a given level.

Wavelet bases like this have been constructed in [CDV] and [AHJP]. Note that Ciesielski's spline bases (see [C], [CD]) can also be used as old bases.

• **Construction of a new system (basis) on** $\Omega = [0, 1]$**.** Let N and k be the parameters of the old basis $\mathcal{A}([0, 1])$. Let $M > 1$ and $\varepsilon > 0$.

For each $I \in \cup_{m \ge m_0} \mathcal{D}_m$, we select a function $\theta_I \in C^k([0, 1])$ so that

B1.

$$|\psi_I^{(j)}(t) - \theta_I^{(j)}(t)| \le \varepsilon |I|^{-j-1/2} \left(1 + \frac{|t-t_I|}{|I|}\right)^{-M}, \quad t \in [0,1], \quad j = 0, 1, \dots, k,$$

and

B2.

$$\int_0^1 t^\nu \theta_I(t)\, dt = 0, \quad j = 0, 1, \dots, k-1.$$

Also, we select, for each $I \in \mathcal{D}_{m_0}$, a function $\omega_I \in C^k([0,1])$ such that

B3.

$$\|\phi_I^{(j)} - \omega_I^{(j)}\|_{C[0,1]} \le \varepsilon |I|^{-j-1/2}, \quad j = 0, 1, \dots, k.$$

Note that **A1**, **A3**, **B1**, and **B3** yield the following decay property: For j=0, 1, ..., k, we have

$$|\psi_I^{(j)}(t)|, |\theta_I^{(j)}(t)|, |\phi_I^{(j)}(t)|, |\omega_I^{(j)}(t)| \le C |I|^{-j-1/2} \left(1 + \frac{|t-t_I|}{|I|}\right)^{-M}, \quad t \in [0,1], \tag{2.7}$$

where we have the restriction $I \in \mathcal{D}_{m_0}$ when considering ω_I.

Now, we define the new system $\mathcal{B}$ on $[0,1]$ by

$$\mathcal{B} := \mathcal{B}([0,1]) := \{\omega_I\}_{I \in \mathcal{D}_{m_0}} \cup \{\theta_I\}_{I \in \cup_{m \ge m_0} \mathcal{D}_m}.$$

It will be shown in §6 that functions $\theta_I, \omega_I \in C^k([0,1])$ satisfying **B1 - B3** exist. Therefore, new bases on $[0,1]$ exist.

It will be convenient for us to unify the notation of the basis functions from $\mathcal{A}([0,1])$ and $\mathcal{B}([0,1])$ as follows. Since $\#\{\varphi_I\}_{I \in \mathcal{D}_{m_0}} = \#\mathcal{D}_{m_0} = 2^{m_0}$, we can use the set $\mathcal{D}_{-1} \cup \cup_{0 \le m \le m_0 - 1} \mathcal{D}_m$ with $\mathcal{D}_{-1} := \mathcal{D}_0 := \{[0,1]\}$ for reindexing the basis functions $\{\phi_I\}_{I \in \mathcal{D}_{m_0}} \subset \mathcal{A}$ and $\{\omega_I\}_{I \in \mathcal{D}_{m_0}} \subset \mathcal{B}$. We set

$$\{\psi_I\}_{I \in \cup_{-1 \le m \le m_0 - 1} \mathcal{D}_m} := \{\phi_I\}_{I \in \mathcal{D}_{m_0}} \quad \text{and} \quad \{\theta_I\}_{I \in \cup_{-1 \le m \le m_0 - 1} \mathcal{D}_m} := \{\omega_I\}_{I \in \mathcal{D}_{m_0}},$$

where the one-to-one correspondence between $\mathcal{D}_{m_0}$ and $\cup_{-1 \le m \le m_0 - 1} \mathcal{D}_m$ is determined by the natural order among the intervals in $\mathcal{D}_{m_0}$ (from left to right) and in $\cup_{-1 \le m \le m_0 - 1} \mathcal{D}_m$ (as it was explained before).

We denote again

$$\mathcal{D} := \mathcal{D}([0,1]) := \cup_{m \ge -1} \mathcal{D}_m.$$

Thus we conveniently have

$$\mathcal{A}([0,1]) = \{\psi_I\}_{I \in \mathcal{D}} \quad \text{and} \quad \mathcal{B}([0,1]) = \{\theta_I\}_{I \in \mathcal{D}}.$$

We can now summarize that, for $I \in \mathcal{D}_m, m \ge m_0$, θ_I satisfies properties **B1** and **B2** and, by **B3**, we have

B3′.

$$\|\psi_I^{(j)} - \theta_I^{(j)}\|_{C[0,1]} \le \varepsilon |I|^{-j-1/2}, \quad I \in \mathcal{D}_m, \quad -1 \le m \le m_0 - 1, \quad j = 0, 1, \dots, k.$$

• **Construction of new bases of periodic functions.** Clearly one can utilize our small perturbation technique to the construction of new bases in the periodic case. We leave the details of this construction to the reader.

In the next sections we shall show that the new systems introduced above inherit most of the good properties of the old bases, provided the parameters ε, k, and M are properly selected.

3 Basic properties of the new systems

We let $\Omega = \mathbb{R}$ or $\Omega = [0,1]$. Let $\mathcal{B} := \mathcal{B}(\Omega) = \{\theta_I\}_{I\in\mathcal{D}}$, $\mathcal{D} := \mathcal{D}(\Omega)$, be the new system constructed in §2. Since $\mathcal{A} := \mathcal{A}(\Omega)$ is an orthonormal basis for $L_2(\Omega)$, then

$$\theta_I = \sum_{J\in\mathcal{D}} a(I,J)\psi_J \quad \text{with} \quad a(I,J) := \langle \theta_I, \psi_J\rangle, \quad I\in\mathcal{D}, \tag{3.1}$$

where the inner product is defined by $\langle f,g\rangle := \int_\Omega f(t)g(t)\,dt$. We denote the matrix of the coefficients by

$$\mathbf{A} := (a(I,J))_{I,J\in\mathcal{D}}. \tag{3.2}$$

The following lemma shows that $\mathbf{A}$ is very close to the identity matrix. In what follows ε, k, and M will be the parameters of $\mathcal{A}$ and $\mathcal{B}$, see §2.

Lemma 3.1. *Let $k \ge 1$ and $M > k+1$. Then the entries $a(I,J)$ of $\mathbf{A}$ satisfy the following properties:*

$$|a(I,J)| \le C_1\varepsilon \left(\min\left\{\frac{|I|}{|J|}, \frac{|J|}{|I|}\right\}\right)^{k+1/2} \left(1 + \frac{|t_I - t_J|}{\max\{|I|,|J|\}}\right)^{-M}, \ I \ne J, \tag{3.3}$$

and

$$|a(I,I) - 1| \le C_1\varepsilon, \tag{3.4}$$

where $C_1 > 1$ is a constant independent of ε.

Proof. Let first $\Omega = \mathbb{R}$. Let $I,J\in\mathcal{D}$, $I \ne J$, and $|J| \le |I|$. We shall estimate both $|a(I,J)|$ and $|a(J,I)|$ under these conditions. This is sufficient for the proof of (3.3).

Without loss of generality we shall assume that $|I| = 1$ and $t_I = 0$. We have, using the orthogonality of ψ_I and ψ_J,

$$a(I,J) := \int_{\mathbb{R}} \theta_I(t)\psi_J(t)\,dt = \int_{\mathbb{R}} (\theta_I(t) - \psi_I(t))\psi_J(t)\,dt. \tag{3.5}$$

We denote

$$g_I := \theta_I - \psi_I.$$

We use the vanishing moments of ψ_J (see **A2**) and (3.5) to obtain

$$|a(I,J)| \;=\; |\int_{\mathbb{R}} [g_I(t) - \sum_{\nu=0}^{k-1} g_I^{(\nu)}(t_J)(t-t_J)^\nu/\nu!]\psi_J(t)\,dt|$$

$$
\begin{aligned}
&\leq \int_{\mathbb{R}} |g_I(t) - \sum_{\nu=0}^{k-1} g_I^{(\nu)}(t_J)(t-t_J)^{\nu})/\nu!||\psi_J(t)|\,dt \\
&= \int_T + \int_{T^c} =: \mathcal{I}_1 + \mathcal{I}_2,
\end{aligned}
$$

where the integral over $\mathbb{R}$ is split up into integrals over $T := \{t : |t - t_J| \geq 1\}$ and T^c.

For $\mathcal{I}_1$, we use **B1**, (2.6), and the definition of T to obtain

$$
\begin{aligned}
\mathcal{I}_1 &\leq C\varepsilon|J|^{-1/2} \int_T [(1+|t|)^{-M} + \sum_{\nu=0}^{k-1} |t-t_J|^{\nu}(1+|t_J|)^{-M}] \left(1 + \frac{|t-t_J|}{|J|}\right)^{-M} dt \\
&\leq C\varepsilon|J|^{-1/2} \int_T [(1+|t|)^{-M} + |t-t_J|^{k-1}(1+|t_J|)^{-M}] \left(1 + \frac{|t-t_J|}{|J|}\right)^{-M} dt \\
&\leq C\varepsilon|J|^{-1/2} \int_T (1+|t|)^{-M} \left(1 + \frac{|t-t_J|}{|J|}\right)^{-M} dt \\
&+ C\varepsilon|J|^{-1/2}(1+|t_J|)^{-M} \int_T |t-t_J|^{k-1} \left(1 + \frac{|t-t_J|}{|J|}\right)^{-M} dt \\
&=: \mathcal{I}_{11} + \mathcal{I}_{12}.
\end{aligned}
$$

To estimate $\mathcal{I}_{11}$ we define $U := \{t : |t| \leq |t_J|/2\}$. We shall integrate over $T \cap U$ and $T \cap U^c$ separately. If $t \in U$, then $|t - t_J| \geq |t_J| - |t| \geq |t_J|/2$ and hence

$$
|t-t_J|/|J| \geq \max\{1, |t_J|/2\}/|J| \geq \frac{1}{4}(1+|t_J|)/|J|. \tag{3.6}
$$

If $t \in U^c := \{t : |t| > |t_J|/2\}$, then $1 + |t| \geq \frac{1}{2}(1 + |t_J|)$. Using this and (3.6), we find

$$
\begin{aligned}
\mathcal{I}_{11} &\leq C\varepsilon|J|^{-1/2} \left(\int_{T\cap U} + \int_{T\cap U^c}\right) \\
&\leq C\varepsilon|J|^{-1/2} \Bigg(|J|^M(1+|t_J|)^{-M} \int_{\mathbb{R}} (1+|t|)^{-M} dt \\
&+ (1+|t_J|)^{-M} \int_{|t-t_J|\geq 1} \left(1 + \frac{|t-t_J|}{|J|}\right)^{-M} dt\Bigg) \\
&\leq C\varepsilon|J|^{M-1/2}(1+|t_J|)^{-M}.
\end{aligned}
$$

We have

$$
\begin{aligned}
\mathcal{I}_{12} &\leq C\varepsilon|J|^{k-1/2}(1+|t_J|)^{-M} \int_{|t-t_J|\geq 1} \left(1 + \frac{|t-t_J|}{|J|}\right)^{-M+k-1} dt \\
&\leq C\varepsilon|J|^{M+1/2}(1+|t_J|)^{-M}.
\end{aligned}
$$

The above estimates for $\mathcal{I}_{11}$ and $\mathcal{I}_{12}$ imply

$$
\mathcal{I}_1 \leq C\varepsilon|J|^{M-1/2}(1+|t_J|)^{-M}. \tag{3.7}
$$

For the integral $\mathcal{I}_2$ over $T^c := \{t : |t - t_J| < 1\}$, we have, using Taylor's formula,

$$\mathcal{I}_2 \le C\varepsilon|J|^{-1/2} \int_{T^c} |t - t_J|^k \|g_I^{(k)}\|_{L_\infty(\Delta_t)} \left(1 + \frac{|t - t_J|}{|J|}\right)^{-M} dt, \tag{3.8}$$

where Δ_t is the interval with end points t_J and t. For each $\xi \in \Delta_t$, we have

$$1 + |\xi| \ge 1 + |t_J| - |\xi - t_J| \ge 1 + |t_J| - |t - t_J| \ge |t_J|$$

and hence $1 + |\xi| \ge (1/2)(1 + |t_J|)$. Therefore, by **B1**,

$$\|g_I^{(k)}\|_{L_\infty(\Delta_t)} \le \varepsilon \max_{\xi\in\Delta_t}(1 + |\xi|)^{-M} \le C\varepsilon(1 + |t_J|)^{-M}.$$

We use this in (3.8) to obtain

$$\mathcal{I}_2 \le C\varepsilon|J|^{k-1/2}(1 + |t_J|)^{-M} \int_{|t-t_J|<1} \left(1 + \frac{|t - t_J|}{|J|}\right)^{-M+k} dt \le C\varepsilon|J|^{k-1/2}(1 + |t_J|)^{-M}.$$

This estimate and (3.7) yield (3.3) when $I \ne J$, and $|J| \le |I|$.

Let us now estimate $|a(J, I)|$ when $I \ne J$ and $|J| \le |I|$. As in (3.5), we have

$$\begin{aligned} a(J, I) &= \int_{\mathbb{R}} (\theta_J(t) - \psi_J(t))\psi_I(t)\, dt \\ &= \int_{\mathbb{R}} [\psi_I(t) - \sum_{\nu=0}^{k-1} \psi_I^{(\nu)}(t_J)(t - t_J)^\nu/\nu!](\theta_J(t) - \psi_J(t))\, dt, \end{aligned}$$

where we used that $\theta_J - \psi_J$ has k vanishing moments, see **A2** and **B2**. We estimate $|a(J, I)|$ using estimate (2.6) for $|\psi_I(t)|$ and $|\psi_I^{(\nu)}(t_J)|$, and **B1** for $|\theta_J(t) - \psi_J(t)|$. Everything else is exactly the same as in the estimate of $|a(I, J)|$ and will be omitted. Thus (3.3) is proved.

To estimate $|a(I, I)|$ we use that $\|\psi_I\|_{L_2(\mathbb{R})} = 1$ and write

$$a(I, I) := \int_{\mathbb{R}} \theta_I(t)\psi_I(t)\, dt = 1 + \int_{\mathbb{R}} (\theta_I(t) - \psi_I(t))\psi_I(t)\, dt.$$

Now, **B1** and estimate (2.6) for ψ_I yield (3.4). Thus (3.3) and (3.4) are proved when $\Omega = \mathbb{R}$.

Let $\Omega = [0, 1]$. If $I, J \in \cup_{m>m_0}\mathcal{D}_m$, then $|a(I, J)|$ and $|a(J, I)|$ can be estimated exactly as in the case $\Omega = \mathbb{R}$ and (3.3) and (3.4) hold.

If $J \in \cup_{m>m_0}\mathcal{D}_m$ and $I \in \cup_{m\le m_0}\mathcal{D}_m$, then in the estimate of $|a(I, J)|$ and $|a(J, I)|$ **B3′** replaces **B1** and everything else is the same as in the case $\Omega = \mathbb{R}$. As a result, (3.3) holds.

If $I, J \in \cup_{m\le m_0}\mathcal{D}_m$ and $I \ne J$, then **B3′** with $j = 0$ yields

$$|a(I, J)|, |a(J, I)| \le C\varepsilon.$$

This estimate implies (3.3) in this case (with C_1 depending on m_0).

If $I \in \cup_{m\le m_0}\mathcal{D}_m$, then (3.4) can be proved exactly as in the case $\Omega = \mathbb{R}$ by using **B3′** instead of **B1**. This completes the proof of Lemma 3.1. □

Let

$$\mathbf{G} := (g(I, J))_{I,J\in\mathcal{D}} := (\langle\theta_I, \theta_J\rangle)_{I,J\in\mathcal{D}} \tag{3.9}$$

be the Gram matrix of $\mathcal{B}$, where $\mathcal{D} = \mathcal{D}(\Omega)$ with $\Omega = \mathbb{R}$ or $\Omega = [0, 1]$. The following lemma shows that G is very close to the identity matrix.

Lemma 3.2. *If $k \geq 1$ and $M > k+1$ then*

$$|g(I,J)| \leq C\varepsilon \left(\min\left\{\frac{|I|}{|J|}, \frac{|J|}{|I|}\right\}\right)^{k+1/2} \left(1 + \frac{|t_I - t_J|}{\max\{|I|,|J|\}}\right)^{-M}, \; I \neq J,\, I, J \in \mathcal{D}, \tag{3.10}$$

and

$$|g(I,I) - 1| \leq C\varepsilon, \quad I \in \mathcal{D}, \tag{3.11}$$

where $C > 1$ is a constant independent of ε.

Proof. Let first $\Omega = \mathbb{R}$. Let $I, J \in \mathcal{D}$ and $I \neq J$. We have

$$|\langle \theta_I, \theta_J \rangle| \leq |\langle \theta_I - \psi_I, \theta_J \rangle| + |\langle \psi_I, \theta_J \rangle| = |\langle \theta_I - \psi_I, \theta_J \rangle| + |a(J, I)|. \tag{3.12}$$

The functions $\{\theta_I\}_{I\in\mathcal{D}}$ as well as $\{\psi_I\}_{I\in\mathcal{D}}$ satisfy the decay conditions (2.6) and have k vanishing moments (see **B2** and **A2**). Also, **B1** holds. Hence $|\langle \theta_I - \psi_I, \theta_J \rangle|$ can be estimated exactly as $|a(I, J)|$ was estimated in the proof of Lemma 3.1. Therefore, the upper bound from (3.3) holds for $|\langle \theta_I - \psi_I, \theta_J \rangle|$. Thus (3.10) holds.

It is readily seen that

$$\begin{aligned} |\langle \theta_I, \theta_I \rangle - 1| &\leq |\langle \theta_I, \theta_I - \psi_I \rangle| + |\langle \theta_I - \psi_I, \psi_I \rangle| + |\langle \psi_I, \psi_I \rangle - 1| \\ &= |\langle \theta_I, \theta_I - \psi_I \rangle| + |\langle \theta_I - \psi_I, \psi_I \rangle|. \end{aligned} \tag{3.13}$$

This, **B1**, and (2.6) yield (3.11).

If $\Omega = [0, 1]$, then we proceed again similarly as in the proof of Lemma 3.1, using (3.12) and (3.13). The details are omitted. Lemma 3.2 is proved. $\square$

We shall next prove that the matrices $\mathbf{A}$ from (3.2) and $\mathbf{G}$ from (3.9) are invertible. This will enable us to proof that the new systems have most of the good properties of the old bases.

We denote by $\ell_\infty^\lambda(\mathcal{D})$ the weighted ℓ_∞ space of all sequences $(c_I)_{I\in\mathcal{D}}$ such that

$$\|(c_I)_{I\in\mathcal{D}}\|_{\ell_\infty^\lambda(\mathcal{D})} := \sup_{I\in\mathcal{D}} |c_I||I|^\lambda < \infty.$$

Theorem 3.1. *Let $k \geq 2$ and $M > k+1$. Then there exists $\varepsilon_0 > 0$ such that for each $0 < \varepsilon \leq \varepsilon_0$ the matrix $\mathbf{A}$ from (3.2) is the matrix of an invertible bounded linear operator $\mathbf{A} : \ell_\infty^\lambda(\mathcal{D}) \to \ell_\infty^\lambda(\mathcal{D})$, $|\lambda| \leq 1/2$. Moreover, $\|\mathbf{Id} - \mathbf{A}\|_{\ell_\infty^\lambda \to \ell_\infty^\lambda} < 1$ and hence $\mathbf{A}^{-1}$ can be defined by its Neumann series*

$$\mathbf{A}^{-1} = \sum_{n=0}^{\infty} (\mathbf{Id} - \mathbf{A})^n \tag{3.14}$$

that is absolutely convergent. In addition to this, the inverse operator $\mathbf{A}^{-1}$ has a matrix

$$\mathbf{A}^{-1} =: (b(I,J))_{I,J\in\mathcal{D}}, \tag{3.15}$$

that satisfies the following property: For any selection of the constants α and β so that $3/2 \leq \alpha < k + 1/2$, $1 < \beta \leq M$, and $2\alpha \geq \beta$,

$$|b(I,J)| \leq C_3\varepsilon \left(\min\left\{\frac{|I|}{|J|}, \frac{|J|}{|I|}\right\}\right)^{\alpha} \left(1 + \frac{|t_I - t_J|}{\max\{|I|,|J|\}}\right)^{-\beta}, \quad I \neq J, \tag{3.16}$$

and

$$|b(I,I)-1| \le C_3\varepsilon, \tag{3.17}$$

where C_3 is a constant independent of ε.

Remark 3.1. *The proof of Theorem 3.1 is fairly standard. To mention some related works we refer the reader to* [FJ], [Mü], and [L]. *However, since we could not find a reference that was good enough for our purposes and for the completeness of the present paper we give the following proof of this theorem.*

For the proof of Theorem 3.1 we need the following lemma.

Lemma 3.3. *Let $\mathbf{M}_i = (\lambda_i(I,J))_{I,J\in\mathcal{D}}, i=1,2$, be two matrices satisfying the following properties, for $I,J\in\mathcal{D}$,*

$$|\lambda_1(I,J)| \le \left(\min\left\{\frac{|I|}{|J|},\frac{|J|}{|I|}\right\}\right)^{\alpha}\left(1+\frac{|t_I-t_J|}{\max\{|I|,|J|\}}\right)^{-\beta} \tag{3.18}$$

and

$$|\lambda_2(I,J)| \le \left(\min\left\{\frac{|I|}{|J|},\frac{|J|}{|I|}\right\}\right)^{\alpha+\delta}\left(1+\frac{|t_I-t_J|}{\max\{|I|,|J|\}}\right)^{-\beta}, \tag{3.19}$$

where $\alpha>1$, $\beta>1$, $\delta>0$, and $2\alpha\ge\beta$. Let

$$\mathbf{M} := \mathbf{M}_1\mathbf{M}_2 =: (\lambda(I,J))_{I,J\in\mathcal{D}}.$$

Then we have

$$|\lambda(I,J)| \le C_2\left(\min\left\{\frac{|I|}{|J|},\frac{|J|}{|I|}\right\}\right)^{\alpha}\left(1+\frac{|t_I-t_J|}{\max\{|I|,|J|\}}\right)^{-\beta}, \quad I,J\in\mathcal{D}, \tag{3.20}$$

where $C_2>1$ is a constant depending only on α, β, and δ.

We give the proof of this lemma in the appendix.

Proof of Theorem 3.1. We shall prove this theorem only when $\Omega=\mathbb{R}$. Let

$$\mathbf{L} := \mathbf{Id}-\mathbf{A} =: (c(I,J))_{I,J\in\mathcal{D}}$$

and let $\mathbf{L}:\ell_\infty^\lambda(\mathcal{D})\to\ell_\infty^\lambda(\mathcal{D})$ denote the operator with matrix $\mathbf{L}$. We shall show that, for sufficiently small ε, $\|\mathbf{L}\|_{\ell_\infty^\lambda\to\ell_\infty^\lambda}<1$.

Let α and β be so that $3/2\le\alpha<k+1/2$, $1<\beta\le M$, and $2\alpha\ge\beta$. Evidently such α and β exist. Lemma 3.1 yields

$$|c(I,J)| \le C_1\varepsilon\left(\min\left\{\frac{|I|}{|J|},\frac{|J|}{|I|}\right\}\right)^{\alpha+\delta}\left(1+\frac{|t_I-t_J|}{\max\{|I|,|J|\}}\right)^{-\beta}, \quad I,J\in\mathcal{D}, \tag{3.21}$$

where $\delta := k+1/2-\alpha>0$. It is readily seen that

$$\|\mathbf{L}\|_{\ell_\infty^\lambda\to\ell_\infty^\lambda} = \sup_{I\in\mathcal{D}}\sum_{J\in\mathcal{D}}|c(I,J)||I|^\lambda|J|^{-\lambda}.$$

We shall use (3.21) and that $|\lambda| \le 1/2$ to estimate $\|\mathbf{L}\|_{\ell^\lambda_\infty \to \ell^\lambda_\infty}$. We have, for a fixed $I \in \mathcal{D}$,

$$\sum_{J\in\mathcal{D}} |c(I,J)||I|^\lambda |J|^{-\lambda} = \sum_{|J|\le|I|} + \sum_{|J|>|I|} =: \sigma_1 + \sigma_2.$$

To estimate σ_1 we set $I_\mu := I + \mu|I|$. Using (3.21), we find

$$\begin{aligned}
\sigma_1 &:= \sum_{|J|\le|I|} |c(I,J)||I|^\lambda |J|^{-\lambda} \\
&= \sum_{j=0}^{\infty} \sum_{\mu\in\mathbb{Z}} \sum_{\substack{J \subset I_\mu \\ |J| = 2^{-j}|I|}} |c(I,J)||I|^\lambda |J|^{-\lambda} \\
&\le \sum_{j=0}^{\infty} \sum_{\mu\in\mathbb{Z}} 2^{j(1+\lambda)} \max\{|c(I,J)| : J \subset I_\mu, |J| = 2^{-j}|I|\} \\
&\le C\varepsilon \sum_{j=0}^{\infty} 2^{-j(\alpha+\delta-3/2)} \sum_{\mu\in\mathbb{Z}} (1+|\mu|)^{-\beta} \\
&\le C\varepsilon,
\end{aligned}$$

where we used that $\alpha \ge 3/2$, $\delta > 0$, and $\beta > 1$.

We next estimate σ_2. For fixed $j \ge 1$, let J_0 be the dyadic interval with the properties: $J_0 \supset I$ and $|J_0| = 2^j|I|$. Write $J_{\mu j} := J_0 + \mu|J_0|$. We estimate now, using (3.21),

$$\begin{aligned}
\sigma_2 &:= \sum_{|J|>|I|} |c(I,J)||I|^\lambda |J|^{-\lambda} \\
&= \sum_{j=1}^{\infty} \sum_{\mu\in\mathbb{Z}} |c(I,J_{\mu j})||I|^\lambda |J|^{-\lambda} \\
&= C\varepsilon \sum_{j=1}^{\infty} 2^{-j(\alpha+\delta-3/2)} \sum_{\mu\in\mathbb{Z}} (1+|\mu|)^{-\beta} \\
&\le C\varepsilon,
\end{aligned}$$

where we used again that $\alpha \ge 3/2$, $\delta > 0$, and $\beta > 1$. Therefore,

$$\sum_{J\in\mathcal{D}} |c(I,J)| = \sigma_1 + \sigma_2 \le C\varepsilon$$

and hence, for sufficiently small $\varepsilon > 0$,

$$\|\mathbf{L}\|_{\ell^\lambda_\infty \to \ell^\lambda_\infty} \le C\varepsilon < 1. \tag{3.22}$$

Therefore, $\mathbf{A}^{-1} : \ell^\lambda_\infty(\mathcal{D}) \to \ell^\lambda_\infty(\mathcal{D})$ exists and (3.14) holds.

Let

$$\mathbf{L}^n =: (c_n(I,J))_{I,J\in\mathcal{D}}.$$

We shall prove that, for $n = 1, 2, \ldots$,

$$|c_n(I,J)| \le (B\varepsilon)^n \left(\min\left\{\frac{|I|}{|J|}, \frac{|J|}{|I|}\right\}\right)^\alpha \left(1 + \frac{|t_I - t_J|}{\max\{|I|,|J|\}}\right)^{-\beta}, \quad I, J \in \mathcal{D}, \tag{3.23}$$

where $B = C_1 C_2$ with C_1 and C_2 from Lemma 3.1 and Lemma 3.3, respectively. We shall carry out the proof of (3.23) by induction in n. From (3.21), it follows that (3.23) holds for $n = 1$. Suppose that (3.23) holds for some $n \ge 1$. Then we apply Lemma 3.3, using (3.21) and (3.23), to conclude that (3.23) holds for $n + 1$. This completes the proof of (3.23).

We are now in a position to prove that (3.16) and (3.17) are valid. It follows, by (3.14) and (3.23), that, for $\varepsilon < 1/B$ and $I \ne J$,

$$\begin{aligned} |b(I,J)| &\le \sum_{n=1}^\infty |c_n(I,J)| \\ &\le \sum_{n=1}^\infty (B\varepsilon)^n \left(\min\left\{\frac{|I|}{|J|}, \frac{|J|}{|I|}\right\}\right)^\alpha \left(1 + \frac{|t_I - t_J|}{\max\{|I|,|J|\}}\right)^{-\beta} \\ &\le \frac{B\varepsilon}{1 - B\varepsilon} \left(\min\left\{\frac{|I|}{|J|}, \frac{|J|}{|I|}\right\}\right)^\alpha \left(1 + \frac{|t_I - t_J|}{\max\{|I|,|J|\}}\right)^{-\beta}. \end{aligned}$$

Therefore, (3.16) holds. The proof of (3.17) is similar. Theorem 3.1 is proved. □

Corollary 3.1. *Let $k \ge 2$, $M > k+1$, and $0 < \varepsilon \le \varepsilon_0$, where ε_0 is from Theorem 3.1. Then the new system $\mathcal{B}(\Omega)$ is related to the old basis $\mathcal{A}(\Omega)$ as follows*

$$\theta_I = \sum_{J \in \mathcal{D}} a(I,J)\psi_J \quad \textit{and} \quad \psi_I = \sum_{J \in \mathcal{D}} b(I,J)\theta_J, \quad I \in \mathcal{D} := \mathcal{D}(\Omega), \tag{3.24}$$

where $a(I,J)$ are from (3.1) and $b(I,J)$ are from (3.15), and both series converge absolutely in $L_p(\Omega)$, $1 \le p \le \infty$. Moreover, $(\|\theta_I\|_{L_p(\Omega)})_{I \in \mathcal{D}} \in \ell_\infty^{\frac{1}{2}-\frac{1}{p}}(\mathcal{D})$.

Proof. It follows by **A1** that $(\|\psi_I\|_{L_p(\Omega)})_{I \in \mathcal{D}} \in \ell_\infty^{\frac{1}{2}-\frac{1}{p}}(\mathcal{D})$ $(1 \le p \le \infty)$. Therefore, since the operator $\mathbf{A} : \ell_\infty^\lambda(\mathcal{D}) \to \ell_\infty^\lambda(\mathcal{D})$ $(|\lambda| \le 1/2)$ is bounded, the first series in (3.24) converges absolutely in $L_p(\Omega)$ and $(\|\theta_I\|_{L_p(\Omega)})_{I \in \mathcal{D}} \in \ell_\infty^{\frac{1}{2}-\frac{1}{p}}(\mathcal{D})$. Similarly, by Theorem 3.1, $\mathbf{A}^{-1} : \ell_\infty^\lambda(\mathcal{D}) \to \ell_\infty^\lambda(\mathcal{D})$ $(|\lambda| \le 1/2)$ is a bounded operator and, since $(\|\theta_I\|_{L_p(\Omega)})_{I \in \mathcal{D}} \in \ell_\infty^{\frac{1}{2}-\frac{1}{p}}(\mathcal{D})$, then the second series in (3.24) converges absolutely in $L_p(\Omega)$ as well. □

Theorem 3.2. *Let $k \ge 2$ and $M > k + 1$. Then there exists $\varepsilon_0 > 0$ such that for each $0 < \varepsilon \le \varepsilon_0$ the matrix $\mathbf{G}$, defined by (3.9), is the matrix of an invertible bounded linear operator $\mathbf{G} : \ell_\infty^\lambda(\mathcal{D}) \to \ell_\infty^\lambda(\mathcal{D})$, $|\lambda| \le 1/2$. Moreover, $\|\mathbf{Id} - \mathbf{G}\|_{\ell_\infty^\lambda \to \ell_\infty^\lambda} < 1$ and hence $\mathbf{G}^{-1}$ can be defined by its Neumann series*

$$\mathbf{G}^{-1} = \sum_{n=0}^\infty (\mathbf{Id} - \mathbf{G})^n$$

that is absolutely convergent. In addition to this, the inverse operator $\mathbf{G}^{-1}$ *has a matrix*

$$\mathbf{G}^{-1} =: (\tilde{g}(I,J))_{I,J\in\mathcal{D}}, \tag{3.25}$$

which satisfies the following property: For any selection of the constants α *and* β *so that* $3/2 \le \alpha < k + 1/2$, $1 < \beta \le M$, *and* $2\alpha \ge \beta$,

$$|\tilde{g}(I,J)| \le C_4\varepsilon \left(\min\left\{\frac{|I|}{|J|}, \frac{|J|}{|I|}\right\}\right)^{\alpha} \left(1 + \frac{|t_I - t_J|}{\max\{|I|,|J|\}}\right)^{-\beta}, \quad I \ne J, \tag{3.26}$$

and

$$|\tilde{g}(I,I) - 1| \le C_4\varepsilon, \tag{3.27}$$

where $C_4 > 1$ *is a constant independent of* ε.

Proof. By Lemma 3.2, the entries $g(I,J)$ of $\mathbf{G}$ satisfy the same inequalities as the entries of $\mathbf{A}$ (see Lemma 3.1). Therefore, Theorem 3.2 follows by the proof of Theorem 3.1. Note that $\mathbf{G}$ is a self-adjoint operator. $\square$

Remark 3.2. *Let* $\ell_p(\mathcal{D})$, $1 \le p \le \infty$, *be the space of all sequences* $(c_I)_{I\in\mathcal{D}}$ *such that*

$$\|(c_I)_{I\in\mathcal{D}}\|_{\ell_p(\mathcal{D})} := (\sum_{I\in\mathcal{D}} |c_I|^p)^{1/p} < \infty$$

with the ℓ_p*-norm replaced by the* sup*-norm if* $p = \infty$. *It can be proved that the operators* $\mathbf{A}$ *and* $\mathbf{G}$ *from Theorem 3.1 and Theorem 3.2, respectively, are bounded and invertible, considered as operators from* $\ell_p(\mathcal{D})$ *onto* $\ell_p(\mathcal{D})$, $1 \le p \le \infty$. *The proof is similar to the proof of Theorem 3.1.*

Corollary 3.2. *Let* $k \ge 2$ *and* $M > k + 1$. *Then for sufficiently small* ε *the new system* $\mathcal{B}(\Omega) = \{\theta_I\}_{I\in\mathcal{D}(\Omega)}$ *with* $\Omega = \mathbb{R}$ *or* $\Omega = [0,1]$ *has a dual* $\tilde{\mathcal{B}}(\Omega) = \{\tilde{\theta}_I\}_{I\in\mathcal{D}(\Omega)}$ *with*

$$\tilde{\theta}_I := \sum_{J\in\mathcal{D}} \tilde{g}(I,J)\theta_J, \tag{3.28}$$

where the series converges absolutely in $L_p(\Omega)$ *and* $\tilde{\theta}_I \in L_p(\Omega)$ *for each* $1 \le p \le \infty$. *Thus*

$$\langle \theta_I, \tilde{\theta}_J\rangle = \delta_{I,J} \quad \text{for} \quad I,J \in \mathcal{D}.$$

Moreover, $(\|\tilde{\theta}_I\|_{L_p(\Omega)})_{I\in\mathcal{D}} \in \ell_\infty^{\frac{1}{2}-\frac{1}{p}}(\mathcal{D})$, $1 \le p \le \infty$.

Proof. We select ε so that $0 < \varepsilon \le \varepsilon_0$, where ε_0 is from Theorem 3.2. By (2.6) or (2.7), it follows that $(\|\theta_I\|_{L_p(\Omega)})_{I\in\mathcal{D}} \in \ell_\infty^{\frac{1}{2}-\frac{1}{p}}(\mathcal{D})$, $1 \le p \le \infty$. We have, by Theorem 3.2, that $\mathbf{G^{-1}} : \ell_\infty^\lambda(\mathcal{D}) \to \ell_\infty^\lambda(\mathcal{D})$ ($|\lambda| \le 1/2$) is a bounded linear operator. Therefore, the series from (3.28) converges absolutely in $L_p(\Omega)$ and hence $\tilde{\theta}_I \in L_p(\Omega)$, $1 \le p \le \infty$ and $(\|\tilde{\theta}_I\|_{L_p(\Omega)})_{I\in\mathcal{D}} \in \ell_\infty^{\frac{1}{2}-\frac{1}{p}}(\mathcal{D})$. Using this, we obtain, for $I,J \in \mathcal{D}$,

$$\langle \theta_J, \tilde{\theta}_I\rangle = \sum_{\Delta\in\mathcal{D}} \tilde{g}(I,\Delta)\langle\theta_J,\theta_\Delta\rangle = \sum_{\Delta\in\mathcal{D}} \tilde{g}(I,\Delta)g(\Delta,J) = \delta_{I,J},$$

since $\mathbf{G^{-1}G} = \mathbf{Id}$. $\square$

4 The new system on $[0, 1]$ is a Schauder basis for $C[0, 1]$

It is our main goal in this section to prove the following theorem.

Theorem 4.1. *If $k \geq 4$ and $M > 5$, then for sufficiently small ε the new system $\mathcal{B} = \mathcal{B}([0, 1])$ from §2 is a Schauder basis for $C[0, 1]$.*

It is easy to prove that a good orthogonal wavelet basis (like the old basis $\mathcal{A}$) is a Schauder basis for $C[0, 1]$ because of the existence of the scaling functions $\{\phi_I\}_{I\in\mathcal{D}_m}$ which span V_m. Our plan is to create a similar structure in the new system $\mathcal{B}$ and use it to prove that $\mathcal{B}$ is a Schauder basis for $C[0, 1]$. Our construction will be based on the following proposition.

Proposition 4.1. *Let $\{\theta_\nu\}_{\nu=1}^\infty$ be a sequence in a Banach space X. Then $\{\theta_\nu\}_{\nu=1}^\infty$ is a Schauder basis of X if and only if the following conditions hold:*

(i) *$\{\theta_\nu\}_{\nu=1}^\infty$ is complete in X (the closed span of $\{\theta_\nu\}_{\nu=1}^\infty$ is the all of X).*

(ii) *There exists a dual system $\{\tilde{\theta}_\nu\}_{\nu=1}^\infty \subset X^*$ ($\langle\theta_\mu, \tilde{\theta}_\nu\rangle = \delta_{\mu,\nu}$).*

(iii) *For each $m \geq m_0$ ($m_0 \geq 0$ is fixed) there exist sets $\{\omega_{\nu,m}\}_{\nu=1}^{2^m} \subset X$ and $\{\tilde{\omega}_{\nu,m}\}_{\nu=1}^{2^m} \cup \{\tilde{\theta}_{\nu,m}\}_{\nu=2^m+1}^\infty \subset X^*$ such that the following conditions hold:*

(a)

$$\text{Span } \{\omega_{\nu,m}\}_{\nu=1}^{2^m} = \text{Span } \{\theta_\nu\}_{\nu=1}^{2^m} =: X_m.$$

(b)

$$\{\tilde{\omega}_{\nu,m}\}_{\nu=1}^{2^m} \cup \{\tilde{\theta}_{\nu,m}\}_{\nu=2^m+1}^\infty \quad \textit{is the dual of} \quad \{\omega_{\nu,m}\}_{\nu=1}^{2^m} \cup \{\theta_\nu\}_{\nu=2^m+1}^\infty.$$

(c) *For each $f \in X$ and $0 \leq i < 2^m$,*

$$\|\sum_{\nu=1}^{2^m} \langle f, \tilde{\omega}_{\nu,m}\rangle\omega_{\nu,m} + \sum_{\nu=2^m+1}^{2^m+i} \langle f, \tilde{\theta}_{\nu,m}\rangle\theta_\nu\| \leq K\|f\|, \tag{4.1}$$

where K is a constant independent of f, m, and i; $\langle f, \tilde{g}\rangle$ denotes the value of the linear functional $\tilde{g} \in X^*$ at f.

Remark 4.1. *We recall that $\{\theta_\nu\}_{\nu=1}^\infty \subset X$ (X a Banach space) is called a Schauder basis for X if for each $f \in X$ there exists a unique sequence of scalars $\{a_\nu\}_{\nu=1}^\infty$ so that $f = \sum_{\nu=1}^\infty a_\nu\theta_\nu$ in X, see, e.g.,* [LT] *or* [KS].

Note that conditions (i) – (iii) *readily imply that $\{\tilde{\omega}_{\nu,m}\}_{\nu=1}^{2^m} \cup \{\tilde{\theta}_\nu\}_{\nu=2^m+1}^\infty$ is the dual of $\{\omega_{\nu,m}\}_{\nu=1}^{2^m} \cup \{\theta_\nu\}_{\nu=2^m+1}^\infty$ (the dual is unique) and hence $\tilde{\theta}_{\nu,m} = \tilde{\theta}_\nu$, $\nu \geq 2^m$. Therefore, in the formulation of Proposition 4.1, $\{\tilde{\theta}_{\nu,m}\}_{\nu=2^m+1}^\infty$ can be replaced by $\{\tilde{\theta}_\nu\}_{\nu=2^m+1}^\infty$.*

We note that Proposition 4.1 is an adaptation to our situation of the standard criterion for checking whether a given sequence is a Schauder basis. Condition (iii) *is usually replaced by the following equivalent condition* (*or a similar one*)*, see, e.g.,* [LT] *or* [KS]:

(iii′) *There exists a constant $K > 0$ such that, for every $f \in X$,*

$$\|\sum_{\nu=1}^{n} \langle f, \tilde{\theta}_\nu\rangle\theta_\nu\| \leq K\|f\|, \quad n = 1, 2, \ldots.$$

For completeness, we give the proof of the part of Proposition 4.1 the we need in the appendix.

We denote

$$\mathcal{E}_m := \bigcup_{\nu=-1}^{m-1} \mathcal{D}_\nu \qquad \text{and} \qquad X_m := \text{Span } \{\theta_I\}_{I\in\mathcal{E}_m},$$

where θ_I are from $\mathcal{B}[0,1]$. Note that $\#\mathcal{E}_m = \dim X_m = 2^m$.

In what follows, we shall assume that $k = 4$ and $M > 5$. Also, we shall assume that the parameters $\alpha, \beta, r, \alpha'$, and β' are selected so that the following inequalities hold: $3/2 \le \alpha < k + 1/2$, $1 < \beta \le M$, $2\alpha \ge \beta$, $\alpha > r + 3/2$, $\alpha + 1/2 > \beta$, $1 \le r \le k$, $\beta > r + 1$, $3/2 < \alpha' < r + 1/2$, $1 < \beta' \le \beta$, and $2\alpha' \ge \beta'$, where r is an integer. These parameters should be selected in the following order: $\beta', \alpha', r, \beta$, and α. Here is one possible selection of the parameters: $\beta' = 1 + \eta$, $\alpha' = 1.5 + \eta$, $r = 2$, $\beta = 3 + \eta$, and $\alpha = 3.5 + \eta$ with $\eta > 0$ sufficiently small. We assume that m_0 is the constant from §2. Also, we shall assume that $\varepsilon > 0$ from the construction of the new system $\mathcal{B}$ in §2 is small enough, namely, so small that Theorem 3.1, Theorem 3.2, Corollary 3.1, and Corollary 3.2 apply.

Lemma 4.1. *For any $m \ge m_0$ and $I \in \mathcal{D}_m$ there exists $\omega_I \in X_m$ such that*

$$|\phi_I^{(j)}(t) - \omega_I^{(j)}(t)| \le C\varepsilon |I|^{-j-1/2} \left(1 + \frac{|t - t_I|}{|I|}\right)^{-\beta}, \; t \in [0,1], \; j = 0, 1, \ldots, \mathrm{r}, \tag{4.2}$$

where C is a constant independent of ε, m, and I.

Proof. If $m = m_0$, then (4.2) follows by **B3** with ω_I defined in **B3** and C depending on m_0.

Let $m > m_0$ and let $I \in \mathcal{D}_m$. We have, by the properties of the old basis,

$$\phi_I = \sum_{J\in\mathcal{E}_m} c(I,J)\psi_J \quad \text{with} \quad c(I,J) := \int_\Omega \phi_I(t)\psi_J(t)\,dt. \tag{4.3}$$

By **A1** and **A3**, we readily find

$$|c(I,J)| \le C|I|^{-1/2}|J|^{-1/2}|I| = C\left(\frac{|I|}{|J|}\right)^{1/2} \quad \text{if} \quad |t_I - t_J| \le 2A|J|,$$

and $c(I,J) = 0$ otherwise. From this, it follows that

$$|c(I,J)| \le C\left(\frac{|I|}{|J|}\right)^{1/2}\left(1 + \frac{|t_I - t_J|}{|J|}\right)^{-\beta}, \quad |J| > |I|, \tag{4.4}$$

with C depending on A and β.

We next approximate each ψ_J, $|J| > |I|$, from the subspace X_m.

Lemma 4.2. *For each $J \in \mathcal{D}_\nu$, $-1 \le \nu \le m-1$, there exists $\lambda_J \in X_m$ such that, for $j = 0, 1, \ldots, \mathrm{r}$,*

$$|\psi_J^{(j)}(t) - \lambda_J^{(j)}(t)| \le C\varepsilon |J|^{-j-1/2} 2^{-(m-\nu)(\alpha-j-1/2)} \left(1 + \frac{|t - t_J|}{|J|}\right)^{-\beta}, \quad t \in [0,1], \tag{4.5}$$

where C is a constant independent of ε, m, J and t.

Proof. We have, by Corollary 3.1,

$$\psi_J = \sum_{\Delta \in \mathcal{D}} b(J,\Delta)\theta_\Delta,$$

where the series converges uniformly. Let

$$\lambda_J := \sum_{\Delta \in \mathcal{E}_m} b(J,\Delta)\theta_\Delta, \quad \lambda_J \in X_m.$$

Then

$$\psi_J - \lambda_J = \sum_{\Delta \in \mathcal{D}\setminus\mathcal{E}_m} b(J,\Delta)\theta_\Delta.$$

Using (2.7) and Theorem 3.1, we obtain, for $j = 0, 1, \ldots, \mathrm{r}$,

$$\begin{aligned}
\sum_{\Delta \in \mathcal{D}\setminus\mathcal{E}_m} \|b(J,\Delta)\theta_\Delta^{(j)}(t)\|_{L_\infty} &\leq C\varepsilon \sum_{\mu=m}^{\infty} \sum_{\Delta \in \mathcal{D}_\mu} \left(\frac{|\Delta|}{|J|}\right)^{\alpha} |\Delta|^{-j-1/2} \\
&\leq C\varepsilon |J|^{-j-1/2} \sum_{\mu=m}^{\infty} \sum_{\Delta \in \mathcal{D}_\mu} \left(\frac{|\Delta|}{|J|}\right)^{\alpha-j-1/2} \\
&\leq C\varepsilon |J|^{-j-1/2} \sum_{\mu=m}^{\infty} 2^{\mu} 2^{-(\mu-m)(\alpha-\mathrm{r}-1/2)} \\
&< \infty,
\end{aligned}$$

since $\alpha > \mathrm{r} + 3/2$. Therefore, the series

$$\sum_{\Delta \in \mathcal{D}\setminus\mathcal{E}_m} b(J,\Delta)\theta_\Delta^{(j)}(t)$$

converges uniformly on $[0,1]$ and hence, for $t \in [0,1]$,

$$\begin{aligned}
|\psi_J^{(j)}(t) - \lambda_J^{(j)}(t)| &\leq \sum_{\Delta \in \mathcal{D}\setminus\mathcal{E}_m} |b(J,\Delta)||\theta_\Delta^{(j)}(t)| \\
&\leq C\varepsilon \sum_{\mu=m}^{\infty} \sum_{\Delta \in \mathcal{D}_\mu} \left(\frac{|\Delta|}{|J|}\right)^{\alpha} \left(1 + \frac{|t_J - t_\Delta|}{|J|}\right)^{-\beta} |\Delta|^{-j-1/2} \left(1 + \frac{|t - t_\Delta|}{|\Delta|}\right)^{-\beta} \\
&\leq C\varepsilon |J|^{-j-1/2} \sum_{\mu=m}^{\infty} 2^{-(\mu-\nu)(\alpha-j-1/2)} \sum_{\Delta \in \mathcal{D}_\mu} \left(1 + \frac{|t_J - t_\Delta|}{|J|}\right)^{-\beta} \left(1 + \frac{|t - t_\Delta|}{|\Delta|}\right)^{-\beta}.
\end{aligned}$$

Applying Lemma 8.1 to the last sum above, we find

$$\begin{aligned}
|\psi_J^{(j)}(t) - \lambda_J^{(j)}(t)| &\leq C\varepsilon |J|^{-j-1/2} \left(1 + \frac{|t - t_J|}{|J|}\right)^{-\beta} \sum_{\mu=m}^{\infty} 2^{-(\mu-\nu)(\alpha-j-1/2)} \\
&\leq C\varepsilon |J|^{-1/2} 2^{-(m-\nu)(\alpha-j-1/2)} \left(1 + \frac{|t - t_J|}{|J|}\right)^{-\beta},
\end{aligned}$$

where we used that $\alpha > \mathrm{r} + 3/2$. Thus (4.5) is proved. $\square$

Completion of the proof of Lemma 4.1. We define, for $I \in \mathcal{D}_m$,

$$\omega_I := \sum_{J\in\mathcal{E}_m} c(I,J)\lambda_J, \quad \omega_I \in X_m,$$

where $c(I,J)$ are from (4.3). Using (4.4) and Lemma 4.2, we find

$$\begin{aligned}
&|\phi_I^{(j)}(t) - \omega_I^{(j)}(t)| \le \sum_{J\in\mathcal{E}_m} |c(I,J)||\psi_J^{(j)}(t) - \lambda_J^{(j)}(t)| \\
\le\ & C\varepsilon \sum_{\nu=-1}^{m-1}\sum_{J\in\mathcal{D}_\nu}\left(\frac{|I|}{|J|}\right)^{1/2}\left(1+\frac{|t_I-t_J|}{|J|}\right)^{-\beta}|J|^{-j-1/2}2^{-(m-\nu)(\alpha-j-1/2)}\left(1+\frac{|t-t_J|}{|J|}\right)^{-\beta} \\
\le\ & C\varepsilon|I|^{-j-1/2}\sum_{\nu=-1}^{m-1}2^{-(m-\nu)(\alpha+1/2)}\sum_{J\in\mathcal{D}_\nu}\left(1+\frac{|t_I-t_J|}{|J|}\right)^{-\beta}\left(1+\frac{|t-t_J|}{|J|}\right)^{-\beta}.
\end{aligned}$$

Applying Lemma 8.1 to the last sum above, we obtain

$$|\phi_I^{(j)}(t) - \omega_I^{(j)}(t)| \le C\varepsilon|I|^{-j-1/2}\sum_{\nu=-1}^{m-1}2^{-(m-\nu)(\alpha+1/2)}\left(1+\frac{|t-t_I|}{2^{-\nu}}\right)^{-\beta}.$$

Clearly

$$1+\frac{|t-t_I|}{2^{-\nu}} \ge 2^{-(m-\nu)}\left(1+\frac{|t-t_I|}{|I|}\right)$$

and hence

$$\begin{aligned}
|\phi_I^{(j)}(t) - \omega_I^{(j)}(t)| &\le C\varepsilon|I|^{-j-1/2}\left(1+\frac{|t-t_I|}{|I|}\right)^{-\beta}\sum_{\nu=-1}^{m-1}2^{-(m-\nu)(\alpha+1/2-\beta)} \\
&\le C\varepsilon|I|^{-j-1/2}\left(1+\frac{|t-t_I|}{|I|}\right)^{-\beta},
\end{aligned}$$

where we used that $\alpha + 1/2 > \beta$. Lemma 4.1 is proved. $\square$

Proof of Theorem 4.1. We shall prove that conditions (i) - (iii) of Proposition 4.1 hold with $X := C[0,1]$, $\{\theta_\nu\}_{\nu=1}^\infty := \{\theta_I\}_{I\in\mathcal{D}}$, $\{\tilde\theta_\nu\}_{\nu=1}^\infty := \{\tilde\theta_I\}_{I\in\mathcal{D}}$, and $\{\omega_{\nu,m}\}_{\nu=1}^{2^m} := \{\omega_I\}_{I\in\mathcal{D}_m}$, $m = 0, 1, \ldots$, where $\tilde\theta_I$ are from Corollary 3.2 and $\omega_I \in X_m$ are from Lemma 4.1.

Since $\mathcal{A}$ is a Schauder basis for $C[0,1]$ (see property **A5** of $\mathcal{A}$) $\mathcal{A}$ is complete in $C[0,1]$. This and Corollary 3.1 imply that $\mathcal{B}$ is complete in $C[0,1]$ as well. Thus condition (i) holds.

By Corollary 3.2, we have that $\tilde{\mathcal{B}} := \{\tilde\theta_I\}_{I\in\mathcal{D}}$ with

$$\tilde\theta_I := \sum_{J\in\mathcal{D}} \tilde g(I,J)\theta_J,\ I \in \mathcal{D}, \tag{4.6}$$

is the dual of $\mathcal{B}$. Thus condition (ii) holds. Note that the dual $\tilde{\mathcal{B}}$ is unique since $\mathcal{B}$ is complete in $C[0,1]$.

It remains to prove that condition (iii) holds. Let $m \geq m_0$. Let $\mathcal{D}_m^0 := \mathcal{D}_m$ ($\mathcal{D}_m^0$ is a copy of $\mathcal{D}_m$) and $\mathcal{D}_m^+ := \cup_{\nu=m}^{\infty} \mathcal{D}_\nu$. We define

$$\mathcal{B}_m := \{\omega_I\}_{I \in \mathcal{D}_m^0} \cup \{\theta_I\}_{I \in \mathcal{D}_m^+},$$

where ω_I are from Lemma 4.1. Let

$$\mathbf{G}_m = (g_m(I,J))_{I,J \in \mathcal{D}_m^0 \cup \mathcal{D}_m^+} \tag{4.7}$$

be the Gram matrix of $\mathcal{B}_m$. We next show that $\mathbf{G}_m$ is close to the identity matrix.

Lemma 4.3. *We have, for $I, J \in \mathcal{D}_m^0 \cup \mathcal{D}_m^+$,*

$$|g_m(I,J)| \leq C\varepsilon \left(\min\left\{ \frac{|I|}{|J|}, \frac{|J|}{|I|} \right\} \right)^{r+1/2} \left(1 + \frac{|t_I - t_J|}{\max\{|I|, |J|\}} \right)^{-\beta}, \quad I \neq J, \tag{4.8}$$

and

$$|g_m(I,I) - 1| \leq C\varepsilon, \tag{4.9}$$

where $C > 1$ is a constant independent of ε.

Proof. If $I, J \in \mathcal{D}_m^+$, then (4.8) and (4.9) follow by Lemma 3.2 ($k \geq r$).

By (2.7) and Lemma 4.1, it follows that, for $I \in \mathcal{D}_m$ and $j = 0, 1, \ldots, r$,

$$|\phi_I^{(j)}(t)|, |\psi_I^{(j)}(t)|, |\omega_I^{(j)}(t)|, |\theta_I^{(j)}(t)| \leq C|I|^{-j-1/2} \left(1 + \frac{|t - t_I|}{|I|} \right)^{-\beta}, \ t \in [0,1]. \tag{4.10}$$

Let $I, J \in \mathcal{D}_m^0$ and $I \neq J$. Then we have

$$\begin{aligned} |g_m(I,J)| &= |\langle \omega_I, \omega_J \rangle| = |\langle \omega_I, \omega_J \rangle - \langle \phi_I, \phi_J \rangle| \\ &\leq |\langle \omega_I, \omega_J - \phi_J \rangle| + |\langle \omega_I - \phi_I, \phi_J \rangle| \\ &\leq \int_0^1 |\omega_I(t)||\omega_J(t) - \phi_J(t)| \, dt + \int_0^1 |\omega_I(t) - \phi_I(t)||\phi_J(t)| \, dt, \end{aligned}$$

where we used that $\langle \phi_I, \phi_J \rangle = 0$. This and Lemma 4.1 (with $j = 0$) yield

$$|g_m(I,J)| \leq C\varepsilon |I|^{-1} \int_{\mathbb{R}} \left(1 + \frac{|t - t_I|}{|I|} \right)^{-\beta} \left(1 + \frac{|t - t_J|}{|I|} \right)^{-\beta} dt \leq C\varepsilon \left(1 + \frac{|t_I - t_J|}{|I|} \right)^{-\beta} \tag{4.11}$$

Hence (4.8) holds for $I, J \in \mathcal{D}_m^0$ and $I \neq J$. Similarly, we obtain, for $I \in \mathcal{D}_m^0$,

$$|g_m(I,I) - 1| = |\langle \omega_I, \omega_I \rangle - 1| \leq C\varepsilon.$$

This is (4.9).

Let $I \in \mathcal{D}_m^0$, $J \in \mathcal{D}_m^+$, and $I \neq J$. We have

$$|g(I,J)| = |\langle \omega_I, \theta_J \rangle| = |\langle \omega_I, \theta_J \rangle - \langle \phi_I, \psi_J \rangle| \leq |\langle \omega_I, \theta_J - \psi_J \rangle| + |\langle \omega_I - \phi_I, \psi_J \rangle|,$$

where we used that $\langle \phi_I, \psi_J \rangle = 0$. We note that ω_I satisfies (4.10), $\theta_J - \psi_J$ satisfies **B1** and has k ($k \geq r$) vanishing moments, by **A2** and **B2**. Also, $\omega_I - \phi_I$ satisfies (4.2) and ψ_J satisfies

(4.10) and has k ($k \geq r$) vanishing moments. Therefore, we can estimate $|\langle \omega_I, \theta_J - \psi_J \rangle|$ and $|\langle \omega_I - \phi_I, \psi_J \rangle|$ exactly as in the proof of Lemma 3.1 and obtain the upper bound (4.8). The roles of M and k are played now by β and r, respectively. We omit the details. Thus Lemma 4.3 is proved. $\square$

Lemma 4.3 enables us to prove that the Gram matrix $\mathbf{G_m}$ is invertible.

Lemma 4.4. *There exists $\varepsilon_1 > 0$ such that for each $0 < \varepsilon \leq \varepsilon_1$ the matrix $\mathbf{G_m}$, defined by (4.7), is the matrix of an invertible bounded linear operator $\mathbf{G_m} : \ell_\infty^\lambda(\mathcal{D}_m^0 \cup \mathcal{D}_m^+) \to \ell_\infty^\lambda(\mathcal{D}_m^0 \cup \mathcal{D}_m^+)$, $|\lambda| \leq 1/2$. Moreover, $\|\mathbf{Id} - \mathbf{G_m}\|_{\ell_\infty^\lambda \to \ell_\infty^\lambda} < 1$ and hence $\mathbf{G_m}^{-1}$ can be defined by its Neumann series*

$$\mathbf{G_m}^{-1} = \sum_{n=0}^{\infty} (\mathbf{Id} - \mathbf{G_m})^n$$

that is absolutely convergent. In addition to this, the inverse operator $\mathbf{G_m}^{-1}$ has a matrix

$$\mathbf{G_m}^{-1} =: (\tilde{g}_m(I,J))_{I,J \in \mathcal{D}_m^0 \cup \mathcal{D}_m^+}, \tag{4.12}$$

that satisfies the properties

$$|\tilde{g}_m(I,J)| \leq C_4 \varepsilon \left(\min\left\{ \frac{|I|}{|J|}, \frac{|J|}{|I|} \right\} \right)^{\alpha'} \left(1 + \frac{|t_I - t_J|}{\max\{|I|,|J|\}} \right)^{-\beta'}, \quad I \neq J, \tag{4.13}$$

and

$$|\tilde{g}_m(I,I) - 1| \leq C_4 \varepsilon, \tag{4.14}$$

where $C_4 > 1$ is a constant independent of ε.

Proof. It is readily seen that Lemma 3.3 holds with $\mathcal{D}$ replaced by $\mathcal{D}_m^0 \cup \mathcal{D}_m^+$ (m fixed) and C_2 independent of m. Now, Lemma 4.4 follows by Lemma 4.3. The proof is almost identical to the proof of Theorem 3.1 (or Theorem 3.2). The roles of M, k, α, and β are played now by β, r, α', and β', respectively. Note that we assumed earlier that the latter parameters satisfy the inequalities $3/2 < \alpha' < r + 1/2$, $1 < \beta' \leq \beta$, and $2\alpha' \geq \beta'$ which replace the corresponding inequalities for α and β from Theorem 3.1 and Theorem 3.2. We leave the details of this proof to the reader. $\square$

Completion of the proof of condition (iii). We define

$$\tilde{\omega}_I := \sum_{J \in \mathcal{D}_m^0} \tilde{g}_m(I,J)\omega_J + \sum_{J \in \mathcal{D}_m^+} \tilde{g}_m(I,J)\theta_J, \quad I \in \mathcal{D}_m^0, \tag{4.15}$$

and

$$\tilde{\theta}_{I,m} := \sum_{J \in \mathcal{D}_m^0} \tilde{g}_m(I,J)\omega_J + \sum_{J \in \mathcal{D}_m^+} \tilde{g}_m(I,J)\theta_J, \quad I \in \mathcal{D}_m^+. \tag{4.16}$$

Exactly as in the proof of Corollary 3.2, it follows, by Lemma 4.4, that

$$\tilde{\mathcal{B}}_m := \{\tilde{\omega}_I\}_{I \in \mathcal{D}_m^0} \cup \{\tilde{\theta}_{I,m}\}_{I \in \mathcal{D}_m^+}.$$

is a dual of $\mathcal{B}_m$. This yields that $\{\omega_I\}_{I\in\mathcal{D}_m^0}$ is a linearly independent set and hence Span $\{\omega_I\}_{I\in\mathcal{D}_m^0} =$ Span $\{\theta_I\}_{I\in\mathcal{E}_m}$. Therefore, $\mathcal{B}_m$ is complete in $C[0,1]$ as well as $\mathcal{B}$. Hence the dual $\tilde{\mathcal{B}}_m$ of $\mathcal{B}_m$ is unique and (see Remark 4.1)

$$\tilde{\theta}_I = \tilde{\theta}_{I,m}, \quad I \in \mathcal{D}_m^+,$$

where $\tilde{\theta}_I$ is from (4.6).

Next, we use Lemma 4.4 and the good localization properties of ω_J and θ_J to show that $\tilde{\omega}_I$ and $\tilde{\theta}_I$ ($I \in \mathcal{D}_m$) have good localization properties. We have, combining (4.10) (with $j = 0$), (4.13), (4.14), and (4.15), for $I \in \mathcal{D}_m^0$,

$$\begin{aligned}
|\tilde{\omega}_I(t) - \omega_I(t)| \;\le\;& |\tilde{g}_m(I,I) - 1||\omega_I(t)| \\
&+ \sum_{J\in\mathcal{D}_m^0,\, J\neq I} |\tilde{g}_m(I,J)||\omega_J(t)| + \sum_{J\in\mathcal{D}_m^+} |\tilde{g}_m(I,J)||\theta_J(t)| \\
\le\;& C\varepsilon|I|^{-1/2} \sum_{J\in\mathcal{D}_m^0} \left(1 + \frac{|t_I - t_J|}{|I|}\right)^{-\beta'} \left(1 + \frac{|t - t_J|}{|I|}\right)^{-\beta'} \\
&+ C\varepsilon|I|^{-1/2} \sum_{J\in\mathcal{D}_m^+} \left(\frac{|J|}{|I|}\right)^{\alpha'-1/2} \left(1 + \frac{|t_I - t_J|}{|I|}\right)^{-\beta'} \left(1 + \frac{|t - t_J|}{|I|}\right)^{-\beta'} \\
=:\;& \sigma_1 + \sigma_2.
\end{aligned}$$

We apply Lemma 8.1 to the sum in σ_1 to find

$$\sigma_1 \le C\varepsilon|I|^{-1/2} \left(1 + \frac{|t - t_I|}{|I|}\right)^{-\beta'}. \tag{4.17}$$

We have

$$\sigma_2 \le C\varepsilon|I|^{-1/2} \sum_{\mu=0}^{\infty} 2^{-\mu(\alpha'-1/2)} \sum_{J\in\mathcal{D}_{m+\mu}} \left(1 + \frac{|t_I - t_J|}{|I|}\right)^{-\beta'} \left(1 + \frac{|t - t_I|}{|I|}\right)^{-\beta'}.$$

Applying Lemma 8.1 to the last sum, we obtain

$$\begin{aligned}
\sigma_2 \;\le\;& C\varepsilon|I|^{-1/2} \left(1 + \frac{|t - t_I|}{|I|}\right)^{-\beta'} \sum_{\mu=0}^{\infty} 2^{-\mu(\alpha'-3/2)} \\
\le\;& C\varepsilon|I|^{-1/2} \left(1 + \frac{|t - t_I|}{|I|}\right)^{-\beta'},
\end{aligned}$$

where we used that $\alpha' > 3/2$. From this and (4.17), it follows that

$$|\tilde{\omega}_I(t) - \omega_I(t)| \le C\varepsilon|I|^{-1/2} \left(1 + \frac{|t - t_I|}{|I|}\right)^{-\beta'}$$

and hence

$$|\tilde{\omega}_I(t)| \le C|I|^{-1/2} \left(1 + \frac{|t - t_I|}{|I|}\right)^{-\beta'}, \quad t \in [0,1],\ I \in \mathcal{D}_m^0. \tag{4.18}$$

Proceeding exactly as above (using (4.10), (4.13), and (4.14) in (4.16)), we obtain

$$|\tilde{\theta}_I(t)| \le C|I|^{-1/2}\left(1+\frac{|t-t_I|}{|I|}\right)^{-\beta'}, \quad t\in[0,1],\ I\in\mathcal{D}_m. \tag{4.19}$$

We are finally completely ready to prove that condition (iii) holds. Let $n = 2^m + i$, where $2^m \le n < 2^{m+1}$ and $0 \le i < 2^m$. Let

$$P_n(f) := \sum_{I\in\mathcal{D}_m^0} \langle f, \tilde{\omega}_I\rangle \omega_I + \sum_{I\in\mathcal{D}_{m,i}^\diamond} \langle f, \tilde{\theta}_I\rangle \theta_I,$$

where $\mathcal{D}_{m,i}^\diamond$ is the set of the first i intervals $I\in\mathcal{D}_m$ (ordered as usual from left to right). We have

$$P_n(f)(t) = \int_0^1 K_n(t,y)f(y)dy,$$

where

$$K_n(t,y) := \sum_{I\in\mathcal{D}_m^0} \tilde{\omega}_I(y)\omega_I(t) + \sum_{I\in\mathcal{D}_{m,i}^\diamond} \tilde{\theta}_I(y)\theta_I(t).$$

We use (4.10) ($\beta > \beta'$), (4.18), and (4.19) to obtain, for $t,y\in[0,1]$,

$$\begin{aligned} |K_n(t,y)| &\le \sum_{I\in\mathcal{D}_m^0} |\tilde{\omega}_I(y)||\omega_I(t)| + \sum_{I\in\mathcal{D}_{m,i}^\diamond} |\tilde{\theta}_I(y)||\theta_I(t)| \\ &\le C|I|^{-1}\sum_{I\in\mathcal{D}_m}\left(1+\frac{|y-t_I|}{|I|}\right)^{-\beta'}\left(1+\frac{|t-t_I|}{|I|}\right)^{-\beta'}. \end{aligned}$$

Applying Lemma 8.1 one more time, we find

$$|K_n(t,y)| \le C|I|^{-1}\left(1+\frac{|t-y|}{|I|}\right)^{-\beta'}, \quad t,y\in[0,1].$$

Therefore, we have, for $f\in C([0,1])$ and $t\in[0,1]$,

$$|P_n(f)(t)| \le \int_0^1 |K_n(t,y)||f(y)|\,dy \le C\|f\|_C|I|^{-1}\int_{\mathbb{R}}\left(1+\frac{|t-y|}{|I|}\right)^{-\beta'}dy \le C\|f\|_C,$$

where we used that $\beta' > 1$. Hence condition (iii) holds and this completes the proof of Theorem 4.1. $\square$

5 The new system is an unconditional basis for L_p, $1 < p < \infty$, and Besov spaces

From the way the new systems were constructed in §2 and their properties from §3, it is clear that they should be unconditional bases for all reasonable spaces of functions of a certain smoothness. In this section we prove this for L_p, $1 < p < \infty$, and Besov spaces, using standard techniques. For the sake of completeness we give the proofs of these results. We shall utilize them to nonlinear n-term approximation in §7.

Theorem 5.1. *If $k \geq 2$ and $M > k+1$, then for sufficiently small $\varepsilon > 0$ the new system $\mathcal{B}(\Omega)$ with $\Omega = \mathbb{R}$ or $\Omega = [0,1]$ (see §2) is an unconditional basis for $L_p(\Omega)$, $1 < p < \infty$.*

Proof. We assume that $\varepsilon > 0$ from the construction of $\mathcal{B}(\Omega)$ is small enough. Namely, let ε be so small that Theorem 3.1 and Corollary 3.1 apply. We select the parameters α and β so that $3/2 \leq \alpha < k + 1/2$, $1 < \beta \leq M$, and $2\alpha \geq \beta$. Therefore, Theorem 3.1 can be used. One possible selection is $\alpha := 1.6$ and $\beta := 1.6$. We shall give the proof only when $\Omega = \mathbb{R}$. We denote briefly $L_p := L_p(\mathbb{R})$.

It is well known (see, e.g., [KS], [LT], [W]) that necessary and sufficient condition for $\mathcal{B}$ to be an unconditional basis for L_p, $1 < p < \infty$, is that $\mathcal{B}$ satisfies the following:

(i) $\mathcal{B}$ is complete in L_p (the closed span of $\mathcal{B}$ is the all of L_p).

(ii) For any finite sequence of numbers $(d_I)_{I \in \mathcal{D}}$, we have

$$\| \sum_{I \in \mathcal{D}} d_I \theta_I \|_{L_p} \approx \| (\sum_{I \in \mathcal{D}} |d_I \theta_I|^2)^{1/2} \|_{L_p} \tag{5.1}$$

with constants of equivalence depending at most on p. Here and later $A \approx B$ means that there are two constants $C_1, C_2 > 0$ such that $C_1 B \leq A \leq C_2 B$.

The completeness of $\mathcal{B}$ in L_p follows by the completeness of $\mathcal{A}$ and Corollary 3.1.

To prove that condition (ii) holds we shall use the Fefferman-Stein vector valued maximal inequality, see [FS]: If $1 < p < \infty$ and $1 < q \leq \infty$, then

$$\| (\sum_{j=1}^{\infty} |M(f_j)|^q)^{1/q} \|_{L_p} \leq C(p,q) \| (\sum_{j=1}^{\infty} |f_j|^q)^{1/q} \|_{L_p} \tag{5.2}$$

with

$$M(f)(t) := \sup_{Q \ni t} |Q|^{-1} \int_Q |f(x)|\, dx,$$

where the sup is taken over all intervals Q containing t.

By the properties of ψ_I and θ_I (**A1, A3, B1**, and $\|\psi\|_{L_2} = 1$), we obtain

$$M(\theta_I)(t) \approx M(\psi_I)(t) \approx M(\lambda_I)(t) \approx |I|^{-1/2} \left(1 + \frac{|t - t_I|}{|I|} \right)^{-1}, \tag{5.3}$$

where $\lambda_I := |I|^{-1/2} \chi_I$ is the characteristic function of I normalized in L_2. By (5.2) and (5.3), it follows that, for any sequence $(d_I)_{I \in \mathcal{D}}$,

$$\| (\sum_{I \in \mathcal{D}} |d_I \psi_I|^2)^{1/2} \|_{L_p} \approx \| (\sum_{I \in \mathcal{D}} |d_I \theta_I|^2)^{1/2} \|_{L_p} \approx \| (\sum_{I \in \mathcal{D}} |d_I \lambda_I|^2)^{1/2} \|_{L_p}. \tag{5.4}$$

Let $(d_I)_{I \in \mathcal{D}}$ be a finite sequence of numbers. We denote

$$f := \sum_{I \in \mathcal{D}} d_I \theta_I.$$

Since $\mathcal{A}$ is an unconditional basis for L_p (see **A4**), then

$$f = \sum_{I \in \mathcal{D}} c_I \psi_I \quad \text{with} \quad c_I = \int_{\mathbb{R}} f(t) \psi_I(t)\, dt$$

and

$$\|f\|_{L_p} \approx \|(\sum_{I\in\mathcal{D}} |c_I\psi_I|^2)^{1/2}\|_{L_p}. \tag{5.5}$$

Note that

$$|c_I| \le \|f\|_{L_p}\|\psi\|_{L_{p'}} \le \|f\|_{L_p}|I|^{1/p'-1/2}, \quad I\in\mathcal{D}, \quad 1/p+1/p'=1,$$

and hence $(c_I)_{I\in\mathcal{D}} \in \ell_\infty^{1/2-1/p'}(\mathcal{D})$. Using Corollary 3.1 and Theorem 3.1, we find

$$c_J = \sum_{I\in\mathcal{D}} a(I,J)d_I \quad \text{with} \quad a(I,J) = \int_{\mathbb{R}} \theta_I(t)\psi_J(t)\,dt, \quad J\in\mathcal{D}, \tag{5.6}$$

and

$$d_J = \sum_{I\in\mathcal{D}} b(I,J)c_I, \quad J\in\mathcal{D}. \tag{5.7}$$

In order to prove (5.1) it is sufficient to prove that

$$\|(\sum_{I\in\mathcal{D}} |c_I\lambda_I|^2)^{1/2}\|_{L_p} \approx \|(\sum_{I\in\mathcal{D}} |d_I\lambda_I|^2)^{1/2}\|_{L_p}. \tag{5.8}$$

Indeed, if (5.8) holds then, using (5.4) and (5.5), we obtain

$$\begin{aligned}\|f\|_{L_p} &\approx \|(\sum_{I\in\mathcal{D}} |c_I\psi_I|^2)^{1/2}\|_{L_p} \approx \|(\sum_{I\in\mathcal{D}} |c_I\lambda_I|^2)^{1/2}\|_{L_p}\\ &\approx \|(\sum_{I\in\mathcal{D}} |d_I\lambda_I|^2)^{1/2}\|_{L_p} \approx \|(\sum_{I\in\mathcal{D}} |d_I\theta_I|^2)^{1/2}\|_{L_p}.\end{aligned}$$

Thus (5.1) holds.

To prove (5.8) we shall use the properties of $a(I,J)$ and $b(I,J)$ from Lemma 3.1 and Theorem 3.1. We shall only prove that

$$\|(\sum_{I\in\mathcal{D}} |d_I\lambda_I|^2)^{1/2}\|_{L_p} \le C\|(\sum_{I\in\mathcal{D}} |c_I\lambda_I|^2)^{1/2}\|_{L_p}. \tag{5.9}$$

The inverse estimate follows in the same way.

The next lemma will help us in the use of the maximal function. We borrowed the idea for this lemma from [FJ].

Lemma 5.1. *Let $J\in\mathcal{D}$, $|J| = 2^{-\nu}$, and $\mu\in\mathbb{Z}$. Let $(h_\Delta)_{\Delta\in\mathcal{D}_\mu}$ be any sequence of numbers. Then the following properties hold.*

(a) *If $\mu\le\nu$, then*

$$\sum_{\Delta\in\mathcal{D}_\mu} |h_\Delta|(1+2^\mu|t_\Delta - t_J|)^{-\beta} \le CM\left(\sum_{\Delta\in\mathcal{D}_\mu} |h_\Delta|\chi_\Delta\right)(t) \quad \textit{for} \quad t\in J. \tag{5.10}$$

(b) *If $\mu>\nu$, then*

$$\sum_{\Delta\in\mathcal{D}_\mu} |h_\Delta|(1+2^\nu|t_\Delta - t_J|)^{-\beta} \le C2^{\mu-\nu}M\left(\sum_{\Delta\in\mathcal{D}_\mu} |h_\Delta|\chi_\Delta\right)(t) \quad \textit{for} \quad t\in J. \tag{5.11}$$

We give the proof of this lemma in the appendix.

Completion of the proof of Theorem 5.1. We have, using (5.7) and Theorem 3.1,

$$\begin{aligned}
&\|(\sum_{J\in\mathcal{D}}|d_J\lambda_J|^2)^{1/2}\|_{L_p}\le\|(\sum_{J\in\mathcal{D}}(\sum_{\Delta\in\mathcal{D}}|b(\Delta,J)||c_\Delta|\lambda_J)^2)^{1/2}\|_{L_p}\\
=&\ \|(\sum_{\nu\in\mathbb{Z}}\sum_{J\in\mathcal{D}_\nu}(\sum_{\mu\le\nu}\sum_{\Delta\in\mathcal{D}_\mu}\cdots+\sum_{\mu>\nu}\sum_{\Delta\in\mathcal{D}_\mu}\cdots)^2)^{1/2}\|_{L_p}\\
\le&\ C\left\|\left(\sum_{\nu\in\mathbb{Z}}\sum_{J\in\mathcal{D}_\nu}\left(\sum_{\mu\le\nu}\sum_{\Delta\in\mathcal{D}_\mu}\left(\frac{|J|}{|\Delta|}\right)^{\alpha-1/2}\left(1+\frac{|t_\Delta-t_J|}{|\Delta|}\right)^{-\beta}|c_\Delta||\Delta|^{-1/2}\chi_J\right)^2\right)^{1/2}\right\|_{L_p}\\
+&\ C\left\|\left(\sum_{\nu\in\mathbb{Z}}\sum_{J\in\mathcal{D}_\nu}\left(\sum_{\mu>\nu}\sum_{\Delta\in\mathcal{D}_\mu}\left(\frac{|\Delta|}{|J|}\right)^{\alpha+1/2}\left(1+\frac{|t_\Delta-t_J|}{|J|}\right)^{-\beta}|c_\Delta||\Delta|^{-1/2}\chi_J\right)^2\right)^{1/2}\right\|_{L_p}\\
=:&\ \sigma_1+\sigma_2.
\end{aligned}$$

We first estimate σ_1. We use Lemma 5.1 and (5.2) to find

$$\begin{aligned}
\sigma_1\ \le&\ C\|(\sum_{\nu\in\mathbb{Z}}\sum_{J\in\mathcal{D}_\nu}(\sum_{\mu\le\nu}2^{-(\nu-\mu)(\alpha-1/2)}\sum_{\Delta\in\mathcal{D}_\mu}(1+2^\mu|t_\Delta-t_J|)^{-\beta}|c_\Delta||\Delta|^{-1/2}\chi_J)^2)^{1/2}\|_{L_p}\\
\le&\ C\|(\sum_{\nu\in\mathbb{Z}}\sum_{J\in\mathcal{D}_\nu}[\sum_{\mu\le\nu}2^{-(\nu-\mu)(\alpha-1/2)}M(\sum_{\Delta\in\mathcal{D}_\mu}|c_\Delta||\Delta|^{-1/2}\chi_\Delta)\chi_J]^2)^{1/2}\|_{L_p}\\
=&\ C\|(\sum_{\nu\in\mathbb{Z}}[\sum_{\mu\le\nu}2^{-(\nu-\mu)(\alpha-1/2)}M(\sum_{\Delta\in\mathcal{D}_\mu}|c_\Delta||\Delta|^{-1/2}\chi_\Delta)]^2)^{1/2}\|_{L_p}\\
\le&\ C\|(\sum_{\nu\in\mathbb{Z}}[M(\sum_{\Delta\in\mathcal{D}_\mu}|c_\Delta||\Delta|^{-1/2}\chi_\Delta)]^2)^{1/2}\|_{L_p}\\
\le&\ C\|(\sum_{\nu\in\mathbb{Z}}[\sum_{\Delta\in\mathcal{D}_\mu}|c_\Delta||\Delta|^{-1/2}\chi_\Delta]^2)^{1/2}\|_{L_p}\\
\le&\ C\|(\sum_{\Delta\in\mathcal{D}}(|c_\Delta|\lambda_\Delta)^2)^{1/2}\|_{L_p},
\end{aligned}$$

where we used the following inequality (see, e.g., [DL]):

$$\sum_{\nu\in\mathbb{Z}}(\sum_{\mu\le\nu}2^{-(\nu-\mu)\delta}a_\mu)^2\le C\sum_{\mu\in\mathbb{Z}}a_\mu^2,\quad \delta>0. \tag{5.12}$$

We now estimate σ_2. Using again Lemma 5.1 and (5.2), we get

$$\begin{aligned}
\sigma_2\ \le&\ C\|(\sum_{\nu\in\mathbb{Z}}\sum_{J\in\mathcal{D}_\nu}(\sum_{\mu>\nu}2^{-(\mu-\nu)(\alpha+1/2)}\sum_{\Delta\in\mathcal{D}_\mu}(1+2^\mu|t_\Delta-t_J|)^{-\beta}|c_\Delta||\Delta|^{-1/2}\chi_J)^2)^{1/2}\|_{L_p}\\
\le&\ C\|(\sum_{\nu\in\mathbb{Z}}\sum_{J\in\mathcal{D}_\nu}[\sum_{\mu>\nu}2^{-(\mu-\nu)(\alpha+1/2)}2^{\mu-\nu}M(\sum_{\Delta\in\mathcal{D}_\mu}|c_\Delta||\Delta|^{-1/2}\chi_\Delta)\chi_J]^2)^{1/2}\|_{L_p}
\end{aligned}$$

$$
\begin{aligned}
&= C\|(\sum_{\nu\in\mathbb{Z}}[\sum_{\mu>\nu} 2^{-(\mu-\nu)(\alpha-1/2)} M(\sum_{\Delta\in\mathcal{D}_\mu} |c_\Delta||\Delta|^{-1/2}\chi_\Delta)]^2)^{1/2}\|_{L_p} \\
&\le C\|(\sum_{\nu\in\mathbb{Z}}[M(\sum_{\Delta\in\mathcal{D}_\mu} |c_\Delta||\Delta|^{-1/2}\chi_\Delta)]^2)^{1/2}\|_{L_p} \\
&\le C\|(\sum_{\nu\in\mathbb{Z}}[\sum_{\Delta\in\mathcal{D}_\mu} |c_\Delta||\Delta|^{-1/2}\chi_\Delta]^2)^{1/2}\|_{L_p} \\
&\le C\|(\sum_{\Delta\in\mathcal{D}} |c_\Delta\lambda_\Delta|^2)^{1/2}\|_{L_p},
\end{aligned}
$$

where we used again inequality (5.12). The above estimates for σ_1 and σ_2 imply (5.9). Theorem 5.1 is proved. □

The Besov spaces are usually defined by moduli of smoothness (see [Pee], [Me], [DL]): $f \in B_q^s(L_p(\mathbb{R}))$, $s > 0$, $0 < p, q \le \infty$, if f is Lebesgue measurable and

$$
|f|_{B_q^s(L_p(\mathbb{R}))} := \left(\int_0^\infty (t^{-s}\omega_k(f,t)_p)^q \,\frac{dt}{t}\right)^{1/q} < \infty,
$$

where the L_q-norm is replaced by the sup-norm if $q = \infty$; $\omega_k(f,t)_p$ is the kth modulus of smoothness of f in $L_p(\mathbb{R})$, $k > s+1$. We fix $k := [s] + 2$. Note that in the above definition f is not necessarily in $L_p(\mathbb{R})$, however, $\Delta_h^k f \in L_p(\mathbb{R})$ for every $h \in \mathbb{R}$. The Besov spaces on $[0,1]$ are defined similarly. For the sake of simplicity we consider in the present paper only Besov spaces on $\mathbb{R}$ which are embedded (modulo polynomials of degree $< k$) in $L_{1+\eta}$ for some $\eta > 0$. Besov spaces like these are needed in nonlinear approximation (see §7.2).

The following characterization of the Besov spaces holds (see [Me], [FJW], [De], [K]). Let $\mathcal{A} = \{\psi_I\}_{I\in\mathcal{D}}$ be an orthonormal wavelet basis (like the old basis from §2) so that the mother wavelet ψ is compactly supported; $\psi \in B_q^\tau(L_p(\mathbb{R}))$ for some $\tau > s$; ψ has k vanishing moments with k as above, and $s > (1/p - 1)_+$. Then $B_q^s(L_p(\mathbb{R}))$ is embedded in $L_{1+\eta}(\mathbb{R})$ for some $\eta > 0$ modulo polynomials of degree $< k$. If $f \in B_q^s(L_p(\mathbb{R}))$, then there exists a polynomial P of degree $< k$ such that

$$
f - P = \sum_{I\in\mathcal{D}} c_I\psi_I \ \text{ in } L_{1+\eta} \text{ with } \ c_I := \textstyle\int_{\mathbb{R}} f(t)\psi_I(t)\,dt \tag{5.13}
$$

and

$$
|f|_{B_q^s(L_p(\mathbb{R}))} \approx (\sum_{m\in\mathbb{Z}}[\sum_{I\in\mathcal{D}_m}(|I|^{-s-1/2+1/p}|c_I|)^p]^{q/p})^{1/q} \tag{5.14}
$$

with the usual modification when $q = \infty$. In describing the convergence of the series from (5.13) and the series that will occur later in this section, we should specify the ordering. We do not do this because all our function series are unconditionally convergent ($\mathcal{A}$ and $\mathcal{B}$ are unconditional bases for L_p, $1 < p < \infty$) and all our series of scalars are absolutely convergent.

In the following theorem we show that the new systems are unconditional bases for Besov spaces.

Theorem 5.2. *Let $0 < p, q \le \infty$ and $s > (1/p-1)_+$. If $k > 2s+2$ and $M > k+1$ then, for sufficiently small $\varepsilon > 0$, the new system $\mathcal{B}(\mathbb{R})$ (see §2) satisfies the following: For every $f \in B^s_q(L_p(\mathbb{R}))$ there exists a polynomial P of degree $< k$ such that*

$$f - P = \sum_{I\in\mathcal{D}} d_I\theta_I \quad \textit{in} \quad L_{1+\eta}(\mathbb{R}),\ \textit{for some } \eta > 0,$$

and

$$|f|_{B^s_q(L_p(\mathbb{R}))} \approx \Big(\sum_{m\in\mathbb{Z}}\Big(\sum_{I\in\mathcal{D}_m}(|I|^{-s-1/2+1/p}|d_I|)^p\Big)^{q/p}\Big)^{1/q} \tag{5.15}$$

with the ℓ_q-norm replaced by the sup*-norm if $q = \infty$.*

Proof. Let $\varepsilon > 0$ (from the construction of $\mathcal{B}(\mathbb{R})$) be so small that Theorem 3.1, Corollary 3.1, and Corollary 3.2 apply. We first select the parameters α and β so that the following inequalities hold: $3/2 \le \alpha < k+1/2$, $1 < \beta \le M$, $2\alpha \ge \beta$, $\alpha > s+1/2$, $\beta > s+1$, and $\alpha - \beta - s - 1/2 + 1/p > 0$. Here is one possible selection of α and β: $\beta := s+1+\delta$ and $\alpha := 2s+3/2+2\delta$ with $\delta > 0$ small enough.

Let $f \in B^s_q(L_p(\mathbb{R}))$. Then there exists a polynomial P of degree $< k$ such that $f - P \in L_{1+\eta}(\mathbb{R})$ for some $\eta > 0$ and (5.13) and (5.14) hold with ψ_I from $\mathcal{A}$. By Theorem 5.1, $\mathcal{B}$ is an unconditional basis for $L_{1+\eta}(\mathbb{R})$. Therefore, $f-P$ can be represented uniquely in the form

$$f - P = \sum_{I\in\mathcal{D}} d_I\theta_I \ \text{ in } \ L_{1+\eta}(\mathbb{R}),$$

where $d_I = \int_{\mathbb{R}} f(t)\tilde{\theta}_I(t)\,dt$ with $\{\tilde{\theta}_I\}_{I\in\mathcal{D}}$ the dual of $\mathcal{B}$ from Corollary 3.2. By using the same corollary, we have

$$\|\tilde{\theta}_I\|_{L_p} \le C|I|^{-1/2+1/p}, \quad 1 \le p \le \infty.$$

Hence, using Hölder's inequality, we find

$$|d_I| \le \|f\|_{L_{1+\eta}}\|\tilde{\theta}_I\|_{L_{1+1/\eta}} \le \|f\|_{L_{1+\eta}}|I|^{-1/2+\eta/(1+\eta)}, \quad I \in \mathcal{D}.$$

Similarly, we have

$$|c_I| \le \|f\|_{L_{1+\eta}}|I|^{-1/2+\eta/(1+\eta)}, \quad I \in \mathcal{D}.$$

Therefore,

$$(c_I)_{I\in\mathcal{D}} \in \ell_\infty^{\frac{1}{2}-\frac{\eta}{1+\eta}} \quad \text{and} \quad (d_I)_{I\in\mathcal{D}} \in \ell_\infty^{\frac{1}{2}-\frac{\eta}{1+\eta}}. \tag{5.16}$$

By Corollary 3.1, Lemma 3.1, and Theorem 3.1, it follows that

$$c_J = \sum_{I\in\mathcal{D}} a(I,J)d_I \quad \text{and} \quad d_J = \sum_{I\in\mathcal{D}} b(I,J)c_I, \quad J \in \mathcal{D}, \tag{5.17}$$

where both series converge absolutely because of (5.16) and the fact that $\mathbf{A}$ and $\mathbf{A}^{-1}$ are both bounded operators from ℓ^λ_∞ onto ℓ^λ_∞ for $|\lambda| \le 1/2$. The properties of $a(I,J)$ and $b(I,J)$ are quite similar (see Lemma 3.1 and Theorem 3.1). For this reason, we shall prove only that

$$|f|_{B^s_q(L_p(\mathbb{R}))} \le C\Big(\sum_{m\in\mathbb{Z}}\Big(\sum_{I\in\mathcal{D}_m}(|I|^{-s-1/2+1/p}|d_I|)^p\Big)^{q/p}\Big)^{1/q} \tag{5.18}$$

with $0 < p, q < \infty$. The inverse estimate can be proved exactly in the same way.

Using (5.14), (5.17), and Lemma 3.1, we find

$$
\begin{aligned}
|f|^q_{B^s_q(L_p(\mathbb{R}))} &\le C \sum_{m\in\mathbb{Z}} [\sum_{J\in\mathcal{D}_m} (|J|^{-s-1/2+1/p} \sum_{I\in\mathcal{D}} |a(I,J)||d_I|)^p]^{q/p}) \\
\le\ & C \sum_{m\in\mathbb{Z}} \left[\sum_{J\in\mathcal{D}_m} \left(\sum_{I\in\mathcal{D},\, |I|\ge|J|} |J|^{-s-1/2+1/p} \left(\frac{|J|}{|I|}\right)^{\alpha} \left(1+\frac{|t_I-t_J|}{|I|}\right)^{-\beta} |d_I| \right)^p \right]^{q/p} \\
&+ C \sum_{m\in\mathbb{Z}} \left[\sum_{J\in\mathcal{D}_m} \left(\sum_{I\in\mathcal{D},\, |I|<|J|} |J|^{-s-1/2+1/p} \left(\frac{|I|}{|J|}\right)^{\alpha} \left(1+\frac{|t_I-t_J|}{|J|}\right)^{-\beta} |d_I| \right)^p \right]^{q/p} \\
=:\ & \sigma_1 + \sigma_2,
\end{aligned}
$$

where, in applying Lemma 3.1, we used that $\alpha < k + 1/2$ and $\beta \le M$.

We denote $h_I := |I|^{-s-1/2+1/p}|d_I|$ and $\gamma := s + 1/2 - 1/p$.

Case I: $p \ge 1$**.** We first estimate σ_1. We have

$$
\begin{aligned}
\sigma_1 &= C \sum_{m\in\mathbb{Z}} \left[\sum_{J\in\mathcal{D}_m} \left(\sum_{I\in\mathcal{D},\, |I|\ge|J|} \left(\frac{|J|}{|I|}\right)^{\alpha-\gamma} \left(1+\frac{|t_I-t_J|}{|I|}\right)^{-\beta} h_I \right)^p \right]^{q/p} \\
&\le C \sum_{m\in\mathbb{Z}} \left[\sum_{J\in\mathcal{D}_m} \left(\sum_{n\le m} 2^{-(m-n)(\alpha-\gamma)} \sum_{I\in\mathcal{D}_n} \left(1+\frac{|t_I-t_J|}{|I|}\right)^{-\beta} h_I \right)^p \right]^{q/p} \\
&\le C \sum_{m\in\mathbb{Z}} \left[\sum_{n\le m} 2^{-(m-n)(\alpha-\gamma)} \left(\sum_{J\in\mathcal{D}_m} \left(\sum_{I\in\mathcal{D}_n} \left(1+\frac{|t_I-t_J|}{|I|}\right)^{-\beta} h_I \right)^p \right)^{1/p} \right]^{q} \\
&=: C \sum_{m\in\mathbb{Z}} \left[\sum_{n\le m} 2^{-(m-n)(\alpha-\gamma)} S_{m,n} \right]^q,
\end{aligned}
$$

where we used Minkowski's inequality ($p \ge 1$). Since $n \le m$, then

$$
\begin{aligned}
S^p_{m,n} &= \sum_{\Delta\in\mathcal{D}_n} \sum_{J\in\mathcal{D}_m,\, J\subset\Delta} \left(\sum_{I\in\mathcal{D}_n} \left(1+\frac{|t_I-t_J|}{|I|}\right)^{-\beta} h_I \right)^p \\
&\le C 2^{m-n} \sum_{\Delta\in\mathcal{D}_n} \left(\sum_{I\in\mathcal{D}_n} \left(1+\frac{|t_I-t_\Delta|}{|I|}\right)^{-\beta} h_I \right)^p,
\end{aligned}
$$

where we used that

$$
1+\frac{|t_I-t_J|}{|I|} \ge \frac{1}{2}\left(1+\frac{|t_I-t_\Delta|}{|I|}\right) \quad \text{if } J\subset\Delta,\ \Delta\in\mathcal{D}_n.
$$

We now define i by the identity $I =: \Delta + i|\Delta|$. We have

$$
\begin{aligned}
S_{m,n} &\le C 2^{(m-n)/p} (\sum_{\Delta\in\mathcal{D}_n} (\sum_{i\in\mathbb{Z}} (1+|i|)^{-\beta} h_{\Delta+i|\Delta|})^p)^{1/p} \\
&\le C 2^{(m-n)/p} \sum_{i\in\mathbb{Z}} (1+|i|)^{-\beta} (\sum_{\Delta\in\mathcal{D}_n} h^p_{\Delta+i|\Delta|})^{1/p},
\end{aligned}
$$

where we applied Minkowski's inequality. Therefore, since $\beta > 1$,

$$S_{m,n} \leq C 2^{(m-n)/p} (\sum_{\Delta \in \mathcal{D}_n} h_\Delta^p)^{1/p}. \tag{5.19}$$

We use this to estimate σ_1. Let $q \geq 1$. Then we have

$$\begin{aligned} \sigma_1 &\leq C(\sum_{m\in\mathbb{Z}}[\sum_{n\leq m} 2^{-(m-n)(\alpha-\gamma-1/p)}(\sum_{\Delta\in\mathcal{D}_n} h_\Delta^p)^{1/p}]^q)^{1/q} \\ &\leq C(\sum_{m\in\mathbb{Z}}[\sum_{\nu=0}^{\infty} 2^{-\nu(\alpha-\gamma-1/p)}(\sum_{\Delta\in\mathcal{D}_{m-\nu}} h_\Delta^p)^{1/p}]^q)^{1/q} \\ &\leq C\sum_{\nu=0}^{\infty} 2^{-\nu(\alpha-\gamma-1/p)}(\sum_{m\in\mathbb{Z}}(\sum_{\Delta\in\mathcal{D}_{m-\nu}} h_\Delta^p)^{q/p})^{1/q}, \end{aligned}$$

where we used again Minkowski's inequality ($q \geq 1$). Therefore,

$$\sigma_1 \leq C(\sum_{n\in\mathbb{Z}}(\sum_{\Delta\in\mathcal{D}_n} h_\Delta^p)^{q/p})^{1/q}, \tag{5.20}$$

where we used that $\alpha - \gamma - 1/p = \alpha - s - 1/2 > 0$.

If $q < 1$, then we use (5.19), the q-triangle inequality $((\sum |y_j|)^q \leq \sum |y_j|^q)$, and change the order of summation to obtain

$$\begin{aligned} \sigma_1 &\leq C\sum_{m\in\mathbb{Z}}\sum_{n\leq m} 2^{-(m-n)(\alpha-\gamma)q} S_{m,n}^q \\ &\leq C\sum_{m\in\mathbb{Z}}\sum_{n\leq m} 2^{-(m-n)(\alpha-\gamma-1/p)q}(\sum_{\Delta\in\mathcal{D}_n} h_\Delta^p)^{q/p} \\ &\leq C\sum_{n\in\mathbb{Z}}\sum_{m\geq n} 2^{-(m-n)(\alpha-\gamma-1/p)q}(\sum_{\Delta\in\mathcal{D}_n} h_\Delta^p)^{q/p} \\ &\leq C\sum_{n\in\mathbb{Z}}(\sum_{\Delta\in\mathcal{D}_n} h_\Delta^p)^{q/p}, \end{aligned}$$

where we used that $\alpha - \gamma - 1/p = \alpha - s - 1/2 > 0$. From this and (5.20) it follows that

$$\sigma_1 \leq C(\sum_{m\in\mathbb{Z}}(\sum_{\Delta\in\mathcal{D}_m}(|I|^{-s-1/2+1/p}|d_I|)^p)^{q/p})^{1/q}, \quad 0 < q < \infty. \tag{5.21}$$

We similarly estimate σ_2. We have

$$\begin{aligned} \sigma_2 &= C\sum_{m\in\mathbb{Z}}\left[\sum_{J\in\mathcal{D}_m}\left(\sum_{I\in\mathcal{D},\,|I|<|J|}\left(\frac{|I|}{|J|}\right)^{\alpha+\gamma}\left(1+\frac{|t_I-t_J|}{|J|}\right)^{-\beta} h_I\right)^p\right]^{q/p} \\ &\leq C\sum_{m\in\mathbb{Z}}\left[\sum_{J\in\mathcal{D}_m}\left(\sum_{n>m} 2^{-(n-m)(\alpha+\gamma)}\sum_{I\in\mathcal{D}_n}\left(1+\frac{|t_I-t_J|}{|J|}\right)^{-\beta} h_I\right)^p\right]^{q/p} \end{aligned}$$

$$\le \; C\sum_{m\in\mathbb{Z}}\left[\sum_{n>m}2^{-(n-m)(\alpha+\gamma)}\left(\sum_{J\in\mathcal{D}_m}\left(\sum_{I\in\mathcal{D}_n}\left(1+\frac{|t_I-t_J|}{|J|}\right)^{-\beta}h_I\right)^p\right)^{1/p}\right]^q$$

$$=: \; C\sum_{m\in\mathbb{Z}}\left[\sum_{n>m}2^{-(n-m)(\alpha+\gamma)}S_{m,n}\right]^q,$$

where we used Minkowski's inequality ($p\ge 1$) as before. Now, we have $n>m$ and hence

$$\begin{aligned} S_{m,n}^p &= \sum_{J\in\mathcal{D}_m}\left(\sum_{\Delta\in\mathcal{D}_m}\sum_{I\in\mathcal{D}_n,\,I\subset\Delta}\left(1+\frac{|t_I-t_J|}{|J|}\right)^{-\beta}h_I\right)^p \\ &\le C\sum_{J\in\mathcal{D}_m}\left[\sum_{\Delta\in\mathcal{D}_m}\left(1+\frac{|t_\Delta-t_J|}{|J|}\right)^{-\beta}\left(\sum_{I\in\mathcal{D}_n,\,I\subset\Delta}h_I\right)\right]^p, \end{aligned}$$

where we used that

$$1+\frac{|t_I-t_J|}{|J|}\ge\frac{1}{2}\left(1+\frac{|t_\Delta-t_J|}{|J|}\right)\quad\text{if } I\subset\Delta,\ \Delta\in\mathcal{D}_m.$$

We now define j by the identity $J=:\Delta+j|\Delta|$. We have

$$\begin{aligned} S_{m,n} &\le C\left(\sum_{J\in\mathcal{D}_m}\left[\sum_{j\in\mathbb{Z}}(1+|j|)^{-\beta}\left(\sum_{I\in\mathcal{D}_n,\,I\subset J-j|J|}h_I\right)\right]^p\right)^{1/p} \\ &\le C\sum_{j\in\mathbb{Z}}(1+|j|)^{-\beta}\left(\sum_{J\in\mathcal{D}_m}\left(\sum_{I\in\mathcal{D}_n,\,I\subset J-j|J|}h_I\right)^p\right)^{1/p} \\ &\le C\sum_{j\in\mathbb{Z}}(1+|j|)^{-\beta}\left(\sum_{J\in\mathcal{D}_m}\left(\sum_{I\in\mathcal{D}_n,\,I\subset J-j|J|}h_I\right)^p\right)^{1/p} \\ &\le C2^{(n-m)(1-1/p)}\Big(\sum_{I\in\mathcal{D}_n}h_I^p\Big)^{1/p}, \end{aligned}$$

where we used Minkowski's and Hölder's (for the last estimate) inequalities. Therefore,

$$S_{m,n}\le C2^{(n-m)(1-1/p)}\Big(\sum_{I\in\mathcal{D}_n}h_I^p\Big)^{1/p}.$$

We use this to estimate σ_2 similarly as we used (5.19) to estimated σ_1. We get the same upper bound for σ_2 as the one for σ_1 from (5.21). We leave the details to the reader. Thus (5.18) holds if $p\ge 1$. Case I is completed.

Case II: $p<1$. We first estimate σ_1. We use the p-triangle inequality and change the order of summation to obtain

$$\sigma_1 \;\le\; C\sum_{m\in\mathbb{Z}}\left[\sum_{J\in\mathcal{D}_m}\sum_{I\in\mathcal{D},\,|I|\ge|J|}\left(\frac{|J|}{|I|}\right)^{(\alpha-\gamma)p}\left(1+\frac{|t_I-t_J|}{|I|}\right)^{-\beta p}h_I^p\right]^{q/p}$$

$$
\begin{aligned}
&\le C\sum_{m\in\mathbb{Z}}\left[\sum_{J\in\mathcal{D}_m}\sum_{n\le m}2^{-(m-n)(\alpha-\gamma)p}\sum_{I\in\mathcal{D}_n}\left(1+\frac{|t_I-t_J|}{|I|}\right)^{-\beta p}h_I^p\right]^{q/p}\\
&\le C\sum_{m\in\mathbb{Z}}\left[\sum_{n\le m}2^{-(m-n)(\alpha-\gamma-\beta)p}\sum_{I\in\mathcal{D}_n}\sum_{J\in\mathcal{D}_m}\left(1+\frac{|t_I-t_J|}{|J|}\right)^{-\beta p}h_I^p\right]^{q/p}\\
&\le C\sum_{m\in\mathbb{Z}}\left[\sum_{n\le m}2^{-(m-n)(\alpha-\gamma-\beta)p}\sum_{I\in\mathcal{D}_n}h_I^p\right]^{q/p},
\end{aligned}
$$

where we used that

$$
1+\frac{|t_I-t_J|}{|I|}\ge\frac{|J|}{|I|}\left(1+\frac{|t_I-t_J|}{|J|}\right),\quad |I|\ge|J|,
$$

and

$$
\sum_{J\in\mathcal{D}_m}\left(1+\frac{|t_I-t_J|}{|J|}\right)^{-\beta p}\le C\sum_{j\imath\mathbb{Z}}(1+|j|)^{-\beta p}\le C,\quad \beta p>\beta/(s+1)>1.
$$

We further estimate σ_1 by using Minkowski's inequality if $q/p\ge 1$ and the q/p-triangle inequality if $q/p<1$. Using that $\alpha-\gamma-\beta=\alpha-\beta-s-1/2+1/p>0$, we obtain that in both cases σ_1 satisfies (5.21).

Finally, we estimate σ_2. We have as above

$$
\begin{aligned}
\sigma_2 &= C\sum_{m\in\mathbb{Z}}\left[\sum_{J\in\mathcal{D}_m}\sum_{I\in\mathcal{D},\,|I|<|J|}\left(\frac{|I|}{|J|}\right)^{(\alpha+\gamma)p}\left(1+\frac{|t_I-t_J|}{|J|}\right)^{-\beta p}h_I^p\right]^{q/p}\\
&\le C\sum_{m\in\mathbb{Z}}\left[\sum_{J\in\mathcal{D}_m}\sum_{n>m}2^{-(n-m)(\alpha+\gamma)p}\sum_{I\in\mathcal{D}_n}\left(1+\frac{|t_I-t_J|}{|J|}\right)^{-\beta p}h_I^p\right]^{q/p}\\
&\le C\sum_{m\in\mathbb{Z}}\left[\sum_{n>m}2^{-(n-m)(\alpha+\gamma)p}\sum_{I\in\mathcal{D}_n}h_I^p\sum_{J\in\mathcal{D}_m}\left(1+\frac{|t_I-t_J|}{|J|}\right)^{-\beta p}\right]^{q/p}\\
&\le C\sum_{m\in\mathbb{Z}}\left[\sum_{n>m}2^{-(n-m)(\alpha+\gamma)p}\sum_{I\in\mathcal{D}_n}h_I^p\right]^{q/p}.
\end{aligned}
$$

We complete the estimate of σ_2 as above and get the upper bound from (5.21). Theorem 5.2 is proved. □

6 Approximation by linear combinations of shifts and dilates of a single function

Let the function ψ satisfy the following properties:

$$
\psi\in C^N(Q), \text{with } Q \text{ an interval so that } 0\in Q \text{ and } |Q|\ge A \ (Q \text{ can be } \mathbb{R}), \tag{6.1}
$$

$$\|\psi^{(j)}\|_{C(Q)} \le C, \quad j = 0, 1, \dots, N, \tag{6.2}$$

$$\text{Supp } \psi \subset [-A, A] \cap Q, \tag{6.3}$$

$$\int_Q t^\nu \psi(t)\, dt = 0, \quad \nu = 0, 1, \dots, k-1, \tag{6.4}$$

and

$$\|\psi\|_{L_2(Q)} = 1, \tag{6.5}$$

where $N \ge k+1$, $k \ge 1$, and $A \ge 1$. The mother wavelet ψ of Daubechies compactly supported and sufficiently smooth wavelets satisfies (6.1) – (6.5).

We are interested in approximating such functions ψ by linear combinations of shifts and dilates of a single smooth and rapidly decaying function. Our general setting is the following: Let $\{\Phi_n\}_{n=1}^\infty$ be a sequence of function with the following properties:

$$\Phi_n \in C^{k+1}(\mathbb{R}), \tag{6.6}$$

$$|\Phi_n^{(j)}(t)| \le \frac{C\, n^{aj+1}}{1 + (n|t|)^M}, \qquad t \in \mathbb{R}, \quad j = 0, 1, \dots, k+1, \tag{6.7}$$

and

$$\int_{\mathbb{R}} \Phi_n(t)\, dt = 1, \tag{6.8}$$

where $M > k$ and k, M, a, and C are independent of n and t.

We now let $\mathbf{\Theta}_K$ denote the set of all functions θ of the form

$$\theta(t) = \sum_{j=1}^{m} a_j \Phi_n(t + b_j) \quad \text{with} \quad mn \le K. \tag{6.9}$$

Theorem 6.1. *Let ψ satisfy* (6.1) – (6.5) *and let Φ_n satisfy* (6.6) – (6.8)*. Then for any $\varepsilon > 0$ there exists a function $\theta \in \mathbf{\Theta}_K$ with K depending on ε and the parameters M, k, A, and C from* (6.1) – (6.8)*, so that*

$$|\psi^{(j)}(t) - \theta^{(j)}(t)| \le \frac{\varepsilon}{(1 + |t|)^M}, \qquad t \in Q, \quad j = 0, 1, \dots, k, \tag{6.10}$$

$$\int_Q t^\nu \theta(t)\, dt = 0, \qquad \nu = 0, 1, \dots, k-1. \tag{6.11}$$

and

$$1 - \varepsilon \le \|\theta\|_{L_2(Q)} \le 1 + \varepsilon. \tag{6.12}$$

Proof. Without loss of generality we shall assume that $A = 1$ and $Q = \mathbb{R}$. We first prove that there exists $\theta \in \mathbf{\Theta}_K$ that satisfies (6.10). We define

$$\lambda(t) := \int_{\mathbb{R}} \psi(t-y)\Phi_n(y)\, dy = \int_{-1}^{1} \psi(u)\Phi_n(t-u)\, dy. \tag{6.13}$$

Using (6.8), we have

$$\psi^{(j)}(t) - \lambda^{(j)}(t) := \int_{\mathbb{R}} (\psi^{(j)}(t) - \psi^{(j)}(t-y))\Phi_n(y)\, dy, \quad j = 0, 1, \ldots, k.$$

If $|t| \le 2$, then we find, using (6.7),

$$\begin{aligned} |\psi^{(j)}(t) - \lambda^{(j)}(t)| &\le \int_{\mathbb{R}} |\psi^{(j)}(t) - \psi^{(j)}(t-y)||\Phi_n(y)|\, dy \\ &\le \|\psi^{(j+1)}\|_C \int_{\mathbb{R}} |y||\Phi_n(y)|\, dy \le C \int_0^\infty \frac{ny}{1+(ny)^M}\, dy \le Cn^{-1}. \end{aligned}$$

If $|t| > 2$, then

$$\begin{aligned} |\psi^{(j)}(t) - \lambda^{(j)}(t)| &\le \int_{\mathbb{R}} |\psi^{(j)}(t-y)||\Phi_n(y)|\, dy = \int_{-1}^{1} |\psi^{(j)}(u)||\Phi_n(t-u)|\, dy \\ &\le C \int_{-1}^{1} |\Phi_n(t-u)|\, dy \le \frac{Cn}{1+(n|t-1|)^M} \le \frac{Cn^{-M+1}}{(1+|t|)^M}. \end{aligned}$$

Therefore, for sufficiently large n,

$$|\psi^{(j)}(t) - \lambda^{(j)}(t)| \le \frac{\varepsilon}{(1+|x|)^M}, \quad t \in \mathbb{R}, \quad j = 0, 1, \ldots, k. \tag{6.14}$$

We now discretize the second integral in the definition of λ from (6.13). To this end we use a very simple quadrature formula. Let

$$\theta_1(t) := \frac{2}{m} \sum_{\mu=1}^{m} \psi(u_\mu)\Phi_n(t-u_\mu) \quad \text{with} \quad u_\mu := -1 + \frac{2\mu}{m}.$$

Note that $\theta_1 \in \mathbf{\Theta}_{mn}$. We have, for $j = 0, 1, \ldots, k$,

$$\begin{aligned} |\lambda^{(j)}(t) - \theta_1^{(j)}(t)| &= |\sum_{\mu=1}^{m} \int_{u_{\mu-1}}^{u_\mu} [\psi(u)\Phi_n^{(j)}(t-u) - \psi(u_\mu)\Phi_n^{(j)}(t-u_\mu)]\, du| \\ &\le \frac{2}{m} \int_{-1}^{1} |\frac{\partial}{\partial u}[\psi(u)\Phi_n^{(j)}(t-u)]|\, du, \end{aligned}$$

where we used the following obvious inequality

$$\int_a^b |f(u) - f(b)|\, du \le (b-a) \int_a^b |f'(u)|\, du, \quad u \in [a,b].$$

Therefore,

$$|\lambda^{(j)}(t) - \theta_1^{(j)}(t)| \le \frac{C}{m}(\|\Phi_n^{(j)}(t-\cdot)\|_{C[-1,1]} + \|\Phi_n^{(j+1)}(t-\cdot)\|_{C[-1,1]}).$$

From this and (6.7), it follows that

$$|\lambda^{(j)}(t) - \theta_1^{(j)}(t)| \le \frac{Cn^{a(k+1)+1}}{m}, \quad |t| \le 2,$$

and

$$|\lambda^{(j)}(t) - \theta_1^{(j)}(t)| \le \frac{Cn^{k+1}}{m(1+(n|t-1|)^M)} \le \frac{Cn^{-M+a(k+1)+1}}{m(1+|t|)^M}, \quad |t| > 2.$$

This and (6.14) yield that, for sufficiently large m,

$$|\psi^{(j)}(t) - \theta_1^{(j)}(t)| \le \frac{2\varepsilon}{(1+|t|)^M}, \quad t \in \mathbb{R}, \quad j = 0, 1, \ldots, k. \tag{6.15}$$

Next, we arrange the needed vanishing moments of θ. Let $\eta := \varepsilon^{-1}(\psi - \theta_1)$. We have, by (6.15),

$$|\eta(t)| \le \frac{2}{(1+|t|)^M}, \quad t \in \mathbb{R}. \tag{6.16}$$

We define

$$w(t) := \begin{cases} \exp(\frac{1}{t^2-1}), & |t| < 1, \\ 0, & |t| \ge 1. \end{cases}$$

Note that $w \in C^\infty(\mathbb{R})$ and Supp $w = [-1, 1]$. We now orthogonalize the powers of t: $1, t, t^2, \ldots$ with respect to the inner product

$$\langle f, g \rangle := \int_{-1}^{1} f(t)g(t)w(t)\, dt.$$

Thus, we obtain a sequence of polynomials $p_0, p_1, p_2, \ldots$ so that $\langle p_\nu, p_\mu \rangle = \delta_{\nu\mu}$ and $p_\nu(t) = \sum_{j=0}^{\nu} a_{\nu j} t^j$ with $a_{\nu\nu} > 0$. Note that

$$\sum_{j=0}^{\nu} a_{\nu j}^2 \le C(k)\|p_\nu\|^2_{L_2[-1/2,1/2]} \le C(k) \int_{-1}^{1} p_\nu^2(t) w(t)\, dt \le C(k). \tag{6.17}$$

Lemma 6.1. *There exist numbers $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ so that*

$$\int_{\mathbb{R}} \left[\eta(t) - \left(\sum_{\mu=0}^{k-1} \alpha_\mu p_\mu(t) \right) w(t) \right] t^\nu\, dt = 0, \quad \nu = 1, 2, \ldots, k-1, \tag{6.18}$$

and

$$|\alpha_\mu| \le C(k, M), \quad \mu = 1, 2, \ldots, k-1. \tag{6.19}$$

Proof. Evidently, (6.18) is equivalent to the following system in $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$:

$$\sum_{\mu=0}^{\nu} \alpha_\mu \langle p_\mu, t^\nu \rangle = \int_{\mathbb{R}} \eta(x) t^\nu \, dt, \quad \nu = 0, 1, \dots, k-1. \tag{6.20}$$

The matrix $\mathbf{A}$ of this system is triangular and the entries along the main diagonal are

$$\langle p_\nu, t^\nu \rangle = \frac{1}{a_{\nu\nu}} \langle p_\nu, p_\nu \rangle = \frac{1}{a_{\nu\nu}}, \quad \nu = 0, 1, \dots, k-1, \quad \text{and hence} \quad \det(\mathbf{A}) = \prod_{\nu=0}^{k-1} a_{\nu\nu}^{-1}.$$

Therefore, using (6.17), we find

$$\det(\mathbf{A}) \geq C(k) > 0.$$

It follows, by (6.17), that $|\langle p_\mu, t^\nu \rangle| \leq C(k)$. On the other hand, by (6.16), we get $|\int_{\mathbb{R}} \eta(t) t^\nu \, dt| \leq C(k, M)$. Therefore, system (6.18) has a unique solution and (6.19) holds. Lemma 6.1 is proved. $\square$

We have, using (6.17),

$$\text{Supp } p_\nu w = [-1, 1] \quad \text{and} \quad \|\frac{d^j}{dt^j}[p_\nu(t) w(t)]\|_{L_\infty(\mathbb{R})} \leq C(k), \quad j = 0, 1, \dots, k. \tag{6.21}$$

Now, by the first part of the proof of the theorem, applied to $p_\nu w$ instead of ψ, it follows that for any $\delta > 0$ there exist functions $g_\nu \in \mathbf{\Theta}_N$, $N = N(\delta)$, so that

$$\left| \frac{d^j}{dt^j}[p_\nu(t) w(t)] - g_\nu^{(j)}(t) \right| \leq \frac{\delta}{(1 + |t|)^M}, \quad j = 0, 1, \dots, k; \; \nu = 0, 1, \dots, k-1. \tag{6.22}$$

Let us consider the following system in $\alpha_0^\star, \alpha_1^\star, \dots, \alpha_{k-1}^\star$:

$$\int_{\mathbb{R}} [\eta(t) - \sum_{\mu=0}^{k-1} \alpha_\mu^\star g_\mu(t)] t^\nu \, dt = 0, \quad \nu = 1, 2, \dots, k-1, \tag{6.23}$$

which is equivalent to the system

$$\sum_{\mu=0}^{k-1} \alpha_\mu^\star \int_{\mathbb{R}} g_\mu(t) t^\nu \, dt = \int_{\mathbb{R}} \eta(t) t^\nu \, dt, \quad \nu = 1, 2, \dots, k-1. \tag{6.24}$$

We have, using (6.22) and that $M > k$,

$$\begin{aligned} \left| \int_{\mathbb{R}} g_\mu(t) t^\nu \, dt - \int_{-1}^{1} p_\mu(t) w(t) t^\nu \, dt \right| &\leq \int_{\mathbb{R}} |g_\mu(t) - p_\mu(t) w(t)| |t|^\nu \, dt \\ &\leq \delta \int_{\mathbb{R}} (1 + |t|)^{-M+k-1} \, dt \leq C\delta. \end{aligned}$$

Therefore, the coefficients of system (6.24) tend to the coefficients of system (6.18) as $\delta \to 0$. From this and Lemma 6.1, it follows that, for sufficiently small δ (depending only on k and M), system (6.24) ((6.23) respectively) has a unique solution $\{\alpha_0^\star, \alpha_1^\star, \dots, \alpha_{k-1}^\star\}$ and

$$|\alpha_\mu^\star| \leq C(k, M), \quad \mu = 0, 1, \dots, k-1. \tag{6.25}$$

We denote

$$\theta_2 := \sum_{\mu=0}^{k-1} \alpha^\star_\mu g_\mu.$$

and let

$$\theta := \theta_1 + \varepsilon\theta_2 \in \mathbf{\Theta}_{mn+kN}.$$

We use (6.23) and the fact that ψ has k vanishing moments to obtain, for $\nu = 1, 2, \ldots, k-1$,

$$\int_{\mathbb{R}} \theta(t)t^\nu\, dt = -\int_{\mathbb{R}} (\psi(t) - \theta_1(t) - \varepsilon\theta_2(t))t^\nu\, dt = -\varepsilon \int_{\mathbb{R}} (\eta(t) - \theta_2(t))t^\nu\, dt = 0.$$

Thus (6.11) holds. Also, we use (6.21) and (6.22) to obtain

$$\begin{aligned} |g_\mu^{(j)}(t)| &\leq \left| \frac{d^j}{dt^j}[p_\nu(t)w(t)] - g_\nu^{(j)}(t) \right| + \left| \frac{d^j}{dt^j}[p_\nu(t)w(t)] \right| \\ &\leq \frac{\delta}{(1+|t|)^M} + \frac{C}{(1+|t|)^M} \leq \frac{C}{(1+|t|)^M}. \end{aligned}$$

From this and (6.25), it follows that

$$|\theta_2^{(j)}(t)| \leq \sum_{\mu=0}^{k-1} |\alpha^\star_\mu||g_\mu^{(j)}(t)| \leq C(k, M) \sum_{\mu=0}^{k-1} |g_\mu^{(j)}(t)| \leq \frac{C}{(1+|t|)^M}, \quad j = 0, 1, \ldots, k,$$

and hence, using also (6.15), we have

$$|\psi^{(j)}(t) - \theta^{(j)}(t)| \leq |\psi^{(j)}(t) - \theta_1^{(j)}(t) - \varepsilon\theta_2^{(j)}(t)| \leq |\psi^{(j)}(t) - \theta_1^{(j)}(t)| + \varepsilon|\theta_2^{(j)}(t)| \leq \frac{C\varepsilon}{(1+|t|)^M}.$$

Thus, θ satisfies (6.10) with ε replaced by $C\varepsilon$ (C independent of ε). So, if we replace the original ε by ε/C, then (6.10) will hold.

We derive (6.12) from (6.5) and (6.10) by replacing again the initial ε by $C\varepsilon$ with C small enough. This completes the proof of Theorem 6.1. $\square$

Remark 6.1. (a) *Clearly, the assumption* Supp $\psi \subset [-A, A] \cap Q$ (*see* (6.3)) *of Theorem 6.1 can be relaxed. It can be replaced by the property*

$$|\psi(t)| \leq \frac{C}{(1+|t|)^S}, \quad t \in \mathbb{R},$$

with S large enough.

(b) *Our method of proving Theorem 6.1 is rude. The point is that we do not need a sophisticated approximation method since we do not have to relate tightly ε and K. The structure of the approximation is important.*

7 Construction of concrete new bases and their application to nonlinear approximation

7.1 Bases generated by a single function. Rational bases.

We want to construct bases $\mathcal{B} = \{\theta_I\}_{I\in\mathcal{D}(\Omega)}$ on $\Omega = \mathbb{R}$ or $\Omega = [0,1]$ with $\{\theta_I\}$ linear combinations of a fixed number of shifts and dilates of a single function Φ_n, namely, $\theta_I \in \mathbf{\Theta}_K$, where $\mathbf{\Theta}_K$ is defined in (6.9). For this, we shall use the construction of bases from §2 and the results from §4 – §6.

Theorem 7.1. *Let $\{\Phi_n\}_{n=1}^{\infty}$ be a sequence of functions satisfying (6.6) – (6.8) with $k \geq 2$ and $M > k+1$. Then, for $\Omega = \mathbb{R}$ and $\Omega = [0,1]$, there exist $K > 0$ and bases $\mathcal{B}(\Omega) = \{\theta_I\}_{I\in\mathcal{D}(\Omega)}$ with $\theta_I \in \mathbf{\Theta}_K$ so that the following properties hold:*
(i) *$\mathcal{B}([0,1])$ is a Schauder basis for $C[0,1]$ provided $k \geq 4$ and $M > 5$.*
(ii) *$\mathcal{B}(\Omega)$ is an unconditional basis for $L_p(\Omega)$, $1 < p < \infty$.*
(iii) *$\mathcal{B}(\mathbb{R})$ is an unconditional basis for the Besov spaces $B_q^s(L_p(\mathbb{R}))$, $0 < p, q \leq \infty$, $s > (1/p-1)_+$, provided $k > 2s+2$ and $M > k+1$ (see Theorem 5.2).*

Proof. This theorem follows immediately by Theorem 4.1, Theorem 5.1, Theorem 5.2, and Theorem 6.1. □

Next, we apply Theorem 7.1 with some concrete functions Φ_n. We denote by $\mathbf{R}_K$ the set of all rational functions of degree K.

Corollary 7.1. *For $\Omega = \mathbb{R}$ and $\Omega = [0,1]$, there exist rational bases $\mathcal{B}_r(\Omega) = \{r_I\}_{I\in\mathcal{D}(\Omega)}$ with $r_I \in \mathbf{R}_K$, K fixed, so that the following properties hold:*
(i) *$\mathcal{B}_r([0,1])$ is a Schauder basis for $C[0,1]$.*
(ii) *$\mathcal{B}_r(\Omega)$ is an unconditional basis for $L_p(\Omega)$, $1 < p < \infty$.*
Moreover, for any $0 < p, q \leq \infty$ and $s > (1/p-1)_+$, $\mathcal{B}_r(\mathbb{R})$ (depending only on s) can be constructed to be an unconditional basis for the Besov spaces $B_q^s(L_p(\mathbb{R}))$ as well.

Proof. Evidently, the rational functions $\Phi_n(t) := C(M)n(1+(nt)^2)^{-[M]}$ with $M > 1$ and $C(M)$ such that $\int_{\mathbb{R}} \Phi_n(t)\,dt = 1$ satisfy (6.6) – (6.8) with $a = 2$ for any k. We fix $M > 5$ and $k \geq 4$. By Theorem 7.1, there exist rational bases $\mathcal{B}_r(\Omega) = \{\theta_I\}_{I\in\mathcal{D}(\Omega)}$ with $\theta_I \in \mathbf{\Theta}_K$, K fixed, that satisfy properties (i) – (ii). In addition to this, if $k > 2s+2$ and $M > k+1$, then (again by Theorem 7.1) $\mathcal{B}_r(\mathbb{R})$ is an unconditional basis for the Besov spaces $B_q^s(L_p(\mathbb{R}))$ as well. Note that $\Phi_n \in \mathbf{R}_{2[M]}$ and hence $\mathbf{\Theta}_K \subset \mathbf{R}_{2[M]K}$. This completes the proof of Corollary 7.1. □

The Gaussian $\Phi_1(t) := \pi^{-1/2}e^{-t^2}$ and its dilates $\Phi_n(t) := \pi^{-1/2}ne^{-(nt)^2}$ is another interesting example of functions satisfying (6.6) – (6.8) ($a = 2$). We denote by $\mathbf{G}_K$ the set of all function g of the form

$$g(t) = \sum_{j=1}^{m} c_j e^{-[n(t+b_j)]^2} \quad \text{with} \quad mn \leq K. \tag{7.1}$$

As above, Theorem 7.1 yields the following.

Corollary 7.2. *For $\Omega = \mathbb{R}$ and $\Omega = [0,1]$, there exist bases $\mathcal{B}_g(\Omega) = \{g_I\}_{I\in\mathcal{D}(\Omega)}$ with $g_I \in \mathbf{G}_K$, K fixed, so that the following properties hold:*

(i) *$\mathcal{B}_g([0,1])$ is a Schauder basis for $C[0,1]$.*

(ii) *$\mathcal{B}_g(\Omega)$ is an unconditional basis for $L_p(\Omega)$, $1 < p < \infty$.*

Moreover, for any $0 < p, q \le \infty$ and $s > (1/p - 1)_+$, $\mathcal{B}_g(\mathbb{R})$ (depending only on s) can be constructed to be an unconditional basis for the Besov spaces $B^s_q(L_p(\mathbb{R}))$ as well.

Another example of functions $\{\Phi_n\}$ satisfying (6.6) – (6.8) is $\Phi_1(t) := (2/\pi)e^t(1+e^{2t})^{-1}$ and its dilates $\Phi_n(t) := \Phi_1(nt)$.

Any reasonable smooth function Φ supported on a compact interval, say $[-1,1]$, and its dilates $\Phi(nt)$ can also play the role of $\{\Phi_n\}$. Here are two examples of Φ's like this:

$$\Phi(t) := \begin{cases} C\exp(\frac{1}{t^2-1}), & |t| < 1, \\ 0, & |t| \ge 1, \end{cases} \qquad \text{and} \qquad \Phi(t) := \mathbf{up}(t) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi t}\prod_{\nu=1}^{\infty}\frac{\sin 2^{-\nu}\xi}{2^{-\nu}\xi}\,d\xi.$$

For more information about the **up**–function, see [RR]. Results similar to the ones from Corollary 7.2 hold for the above selections of functions Φ_n.

7.2 Application of bases to nonlinear approximation

In this part, we show how our bases can be used in nonlinear approximation. We shall restrict our attention to the case $\Omega = \mathbb{R}$.

• **n-term approximation from a basis.** Let $\mathcal{B} = \{\theta_I\}_{I\in\mathcal{D}}$ be a sequence (basis) of functions from $L_p(\mathbb{R})$. We denote by $\Sigma_n := \Sigma_n(\mathcal{B})$ the set of all functions S of the form

$$S = \sum_{I\in\Lambda_n} a_I\theta_I,$$

where $\Lambda_n \subset \mathcal{D}$ and $\#\Lambda_n \le n$. The best n-term approximation of f from $\mathcal{B}$ in the L_p norm is defined by

$$\sigma_n(f,\mathcal{B})_p := \inf_{S\in\Sigma_n}\|f-S\|_{L_p(\mathbb{R})}.$$

Let $A^\gamma_q := A^\gamma_q(L_p,\mathcal{B})$ be the approximation space of all functions f so that

$$|f|_{A^\gamma_q} := (\sum_{n=1}^{\infty}(n^\gamma\sigma_n(f,\mathcal{B})_p)^q\frac{1}{n})^{1/q} < \infty \tag{7.2}$$

with the ℓ_q-norm replaced by the sup-norm if $q = \infty$ as usual. A basic problem of nonlinear approximation is to characterize the approximation spaces A^γ_q. The standard way of doing this is by first proving Jackson and Bernstein inequalities and then using interpolation spaces.

Theorem 7.2. *Let $\mathcal{B} = \{\theta_I\}_{I\in\mathcal{D}(\mathbb{R})}$ be one of the new basis so that $\mathcal{B}$ is an unconditional basis for $L_p(\mathbb{R})$, $1 < p < \infty$, and for the Besov space $B^s_\tau(L_\tau(\mathbb{R}))$ with $s > 0$ and $\frac{1}{\tau} = s + \frac{1}{p}$. Then the following inequalities hold:*

(Jackson inequality) $\quad \sigma_n(f,\mathcal{B})_p \le Cn^{-s}|f|_{B^s_\tau(L_\tau(\mathbb{R}))}, \quad f \in B^s_\tau(L_\tau(\mathbb{R}))\cap L_p(\mathbb{R}),$

(Bernstein inequality) $\quad |S|_{B^s_\tau(L_\tau(\mathbb{R}))} \le Cn^s\|S\|_{L_p(\mathbb{R})}, \quad S \in \Sigma_n.$

Remark 7.1. *We make the assumption* $f \in B^s_\tau(L_\tau(\mathbb{R})) \cap L_p(\mathbb{R})$ *in Theorem 7.2 instead of simply* $f \in B^s_\tau(L_\tau(\mathbb{R}))$ *because* $B^s_\tau(L_\tau(\mathbb{R}))$ *is embedded in* $L_p(\mathbb{R})$ *modulo polynomials of degree* $< k$. *Therefore, we have to eliminate the polynomial that may occur.*

Proof. This theorem follows by Theorem 7.1. The proof can be carried out similarly as the proofs of Theorem 5, Corollary 1, and Theorem 6 from [De]. We leave the details to the reader. □

The Jackson and Bernstein inequalities from Theorem 7.2 imply the following characterization of the approximation spaces A^γ_q (see [DL] or [PP]):

Theorem 7.3. *Let* $1 < p < \infty$ *and* $s > 0$. *We have, for* $0 < \gamma < s$ *and* $0 < q \le \infty$,

$$A^\gamma_q(L_p, \mathcal{B}) = (L_p(\mathbb{R}), B^s_\tau(L_\tau(\mathbb{R})))_{\gamma/s,q}$$

with equivalent norms, where $(X, Y)_{\theta,q}$ *is the real interpolation space between* X *and* Y.

In the particular case of the rational and Gaussian bases from §7.1, we obtain the following.

Corollary 7.3. *Let* $1 < p < \infty$, $s > 0$, *and* $\frac{1}{\tau} = s + \frac{1}{p}$. *Let* $\mathcal{B}$ *be the rational basis* $\mathcal{B}_r(\mathbb{R})$ *from Corollary 7.1 or the Gaussian basis* $\mathcal{B}_g(\mathbb{R})$ *from Corollary 7.2 Then the following inequalities hold:*

$$\text{(J)} \qquad \sigma_n(f, \mathcal{B})_p \le C n^{-s} |f|_{B^s_\tau(L_\tau(\mathbb{R}))}, \quad f \in B^s_\tau(L_\tau(\mathbb{R})) \cap L_p(\mathbb{R}),$$

$$\text{(B)} \qquad |R|_{B^s_\tau(L_\tau(\mathbb{R}))} \le C n^s \|R\|_{L_p(\mathbb{R})}, \quad R \in \Sigma_n(\mathcal{B}).$$

Therefore, for $0 < \gamma < s$ *and* $0 < q \le \infty$,

$$A^\gamma_q(L_p, \mathcal{B}) = (L_p(\mathbb{R}), B^s_\tau(L_\tau(\mathbb{R})))_{\gamma/s,q}.$$

• **Approximation from dictionaries.** We now consider n-term approximation from dictionaries. Dictionaries are collections of functions larger than bases and are redundant. Consider the dictionary $\mathbf{D}$ of all shifts and dilates of a single function Φ. We would like to consider n-term nonlinear approximation from such a dictionary $\mathbf{D}$. We denote by $\mathbf{D}_n$ the set of all functions

$$S = \sum_{j=1}^n a_j \Phi_j, \quad \Phi_j \in \mathbf{D} \ \ (\Phi_j(t) = \Phi(a_j t + b_j)).$$

The best n-term approximation of f from $\mathbf{D}$ in the norm of $L_p(\mathbb{R})$ is defined by

$$\sigma_n(f, \mathbf{D})_p := \inf_{S \in \mathbf{D}_n} \|f - S\|_{L_p(\mathbb{R})}.$$

The approximation spaces $A^\gamma_q(L_p, \mathbf{D})$ are defined similarly as the approximation spaces $A^\gamma_q(L_p, \mathcal{B})$ (see (7.2)). We are interested in characterizing the approximation spaces $A^\gamma_q(L_p, \mathbf{D})$.

A natural problem arises: If there exists a basis $\mathcal{B}$ consisting of functions from $\mathbf{D}$ so that the approximation spaces $A^\gamma_q(L_p, \mathbf{D})$ and $A^\gamma_q(L_p, \mathcal{B})$ are the same? If this happens to be true, then the problem for n-term approximation from $\mathbf{D}$ reduces to the easier problem for n-term approximation from $\mathcal{B}$. In this case, one can use the n-term approximation algorithm discussed in the Introduction. In particular, it is interesting if the bases from this paper could give the desirable bases for some dictionaries $\mathbf{D}$.

Two examples are in order.

(i) Let $\mathbf{R}$ be the dictionary of all shifts and dilates of $r(t) := (1+t^2)^{-1}$. It is easily seen that the rational function $(1+t^2)^{-m}$ can be approximated in L_p $(1 \le p \le \infty)$ with any precision by linear combinations of m dilates of r. Indeed, we have

$$\frac{\partial^{m-1}}{\partial a^{m-1}}(a+t^2)^{-1} = (-1)^{m-1}(m-1)!(a+t^2)^{-m}$$

and hence the $(m-1)$-th differences of $(a+t^2)^{-1}$ in a at 1 will give the approximation we need. Now, let $1 < p < \infty$, $s > 0$, and $\frac{1}{\tau} := s + \frac{1}{p}$. Corollary 7.3 yields

$$\sigma_n(f, \mathbf{R})_p \le \sigma_n(f, \mathcal{B}_r)_p \le Cn^{-s}|f|_{B^s_\tau(L_\tau(\mathbb{R}))}, \quad f \in B^s_\tau(L_\tau(\mathbb{R})) \cap L_p(\mathbb{R}). \tag{7.3}$$

This estimate is equivalent to a result of Pekarskii for rational approximation on $[-1, 1]$, see [Pek1]. In [Pek2], Pekarskii proved the following Bernstein type inequality:

$$\|R^{(s)}\|_{L_\tau[-1,1]} \le Cn^s\|R\|_{L_p[-1,1]}, \quad R \in \mathbf{R}_n \tag{7.4}$$

and used it to relate the rational and spline approximation. By a simple change of variables, these results can be extended on $\mathbb{R}$. Together with (7.3) and Corollary 7.3, they imply that the approximation spaces of $\mathbf{R}$ and $\mathcal{B}_r$ are the same, namely,

$$A^\gamma_q(L_p, \mathbf{R}) = A^\gamma_q(L_p, \mathcal{B}_r) = (L_p(\mathbb{R}), B^s_\tau(L_\tau(\mathbb{R})))_{\gamma/s,q}$$

with equivalent norms provided $1 < p < \infty$, $0 < q \le \infty$, and $0 < \gamma < s$. Therefore, the order of the n-term approximation of a function f in L_p from $\mathbf{R}$ can be achieved by n-term approximation of f from the basis $\mathcal{B}_r$.

(ii) We consider now the same problem for the dictionary $\mathbf{G}$ of all shifts and dilates of the Gaussian. The problem is again whether there exists a basis $\mathcal{B}$ consisting of functions from $\mathbf{G}_K$ with K fixed (see (7.1)) so that the approximation spaces $A^\gamma_q(L_p, \mathbf{G})$ and $A^\gamma_q(L_p, \mathcal{B})$ are the same. The problem would be solved, taking into account Corollary 7.3, if the following Bernstein type inequality holds (an open problem):

$$|G|_{B^s_\tau(L_\tau(\mathbb{R}))} \le Cn^s\|G\|_{L_p(\mathbb{R})},$$

for any function G of the form $G(t) := \sum_{j=1}^n a_j e^{-(b_j t + c_j)^2}$, $a_j, b_j, c_j \in \mathbb{R}$, provided $1 < p < \infty$, $s > 0$, and $\frac{1}{\tau} = s + \frac{1}{p}$.

The same problem for other dictionaries $\mathbf{D}$ seems also interesting.

8 Appendix

8.1 Proof of Lemma 3.3.

We need the following technical lemma for the proof of Lemma 3.3.

Lemma 8.1. *Let m be an integer, $a, b \in \mathbb{R}$, $c > 0$, $d > 0$, and $\beta > 1$. Then the following inequality holds*

$$\sum_{\Delta \in \mathcal{D}_m} \left(1 + \frac{|t_\Delta - a|}{c}\right)^{-\beta} \left(1 + \frac{|t_\Delta - b|}{d}\right)^{-\beta} \leq C(1 + 2^m \min\{c, d\}) \left(1 + \frac{|a - b|}{\max\{c, d\}}\right)^{-\beta}, \tag{8.1}$$

where C is a constant depending only on β.

Proof. We consider only the case when $\mathcal{D}_m = \mathcal{D}_m(\mathbb{R})$. Let $a \leq b$ and $c \leq d$. Let $\delta := 2^{-m}$. Let σ denote the sum from (8.1). We split up σ into two sums as follows

$$\sigma = \sum_{\substack{\Delta \in \mathcal{D}_m \\ |t_\Delta - a| > d}} + \sum_{\substack{\Delta \in \mathcal{D}_m \\ |t_\Delta - a| \leq d}} =: \sigma_1 + \sigma_2.$$

To estimate σ_1 we use that

$$\frac{|t_\Delta - a|}{c} > \frac{d}{2c}\left(1 + \frac{|t_\Delta - a|}{d}\right) \quad \text{when} \quad |t_\Delta - a| > d.$$

We find

$$\begin{aligned}
\sigma_1 &\leq 2^\beta \left(\frac{c}{d}\right)^\beta \sum_{\Delta \in \mathcal{D}_m} \left(1 + \frac{|t_\Delta - a|}{d}\right)^{-\beta} \left(1 + \frac{|t_\Delta - b|}{d}\right)^{-\beta} \\
&\leq 2^\beta \left(\frac{c}{d}\right)^\beta \left(\sum_{\substack{\Delta \in \mathcal{D}_m \\ t_\Delta \leq (a+b)/2}} + \sum_{\substack{\Delta \in \mathcal{D}_m \\ t_\Delta > (a+b)/2}} \right) \\
&\leq 2^\beta \left(\frac{c}{d}\right)^\beta \left(1 + \frac{|a - b|}{2d}\right)^{-\beta} \left(\sum_{\substack{\Delta \in \mathcal{D}_m \\ t_\Delta \leq (a+b)/2}} \left(1 + \frac{|t_\Delta - a|}{d}\right)^{-\beta} \right. \\
&\quad \left. + \sum_{\substack{\Delta \in \mathcal{D}_m \\ t_\Delta > (a+b)/2}} \left(1 + \frac{|t_\Delta - b|}{d}\right)^{-\beta} \right) \\
&\leq C \left(\frac{c}{d}\right)^\beta \left(1 + \frac{|a - b|}{d}\right)^{-\beta} \sum_{\nu=0}^{\infty} \left(1 + \frac{\nu\delta}{d}\right)^{-\beta} \\
&\leq C \left(\frac{c}{d}\right)^\beta \left(1 + \frac{|a - b|}{d}\right)^{-\beta} \left(1 + \int_0^\infty \left(1 + \frac{\delta t}{d}\right)^{-\beta} dt\right).
\end{aligned}$$

Therefore,

$$\sigma_1 \le C\left(\frac{c}{d}\right)^{\beta}\left(1+\frac{d}{\delta}\right)\left(1+\frac{|a-b|}{d}\right)^{-\beta} \le C\left(1+\frac{c}{\delta}\right)\left(1+\frac{|a-b|}{d}\right)^{-\beta}. \tag{8.2}$$

We now estimate σ_2. We shall use that

$$1+\frac{|t_\Delta - b|}{d} \ge \frac{1}{2}\left(1+\frac{|a-b|}{d}\right) \quad \text{when} \quad |t_\Delta - a| \le d.$$

We get

$$\begin{aligned}
\sigma_2 &\le 2^{\beta}\left(1+\frac{|a-b|}{d}\right)^{-\beta}\sum_{\Delta\in\mathcal{D}_m}\left(1+\frac{|t_\Delta - a|}{c}\right)^{-\beta} \\
&\le C\left(1+\frac{|a-b|}{d}\right)^{-\beta}\sum_{\nu=0}^{\infty}\left(1+\frac{\nu\delta}{c}\right)^{-\beta} \\
&\le C\left(1+\frac{|a-b|}{d}\right)^{-\beta}\left(1+\int_0^\infty\left(1+\frac{\delta t}{c}\right)^{-\beta}dt\right) \\
&\le C\left(1+\frac{c}{\delta}\right)\left(1+\frac{|a-b|}{d}\right)^{-\beta}.
\end{aligned}$$

This and (8.2) imply (8.1). Lemma 8.1 is proved. □

Proof of Lemma 3.3. Evidently, it is sufficient to prove the lemma only when $\Omega = \mathbb{R}$. We fix $I, J \in \mathcal{D}$ so that $|I| \ge |J|$. Let $I \in \mathcal{D}_\nu$ and $J \in \mathcal{D}_{\nu+\mu}$, $\mu \ge 0$. Hence $|I| = 2^\mu |J|$. We have

$$\begin{aligned}
|\lambda(I,J)| &\le \sum_{\Delta\in\mathcal{D}}|\lambda_1(I,\Delta)||\lambda_2(\Delta,J)| \\
&= \sum_{|\Delta|<|J|} + \sum_{|J|\le|\Delta|\le|I|} + \sum_{|\Delta|>|I|} \\
&=: \sigma_1 + \sigma_2 + \sigma_3.
\end{aligned}$$

We first estimate σ_1. Using (3.18) and (3.19), we find

$$\begin{aligned}
\sigma_1 &\le \sum_{|\Delta|<|J|}\left(\frac{|\Delta|}{|I|}\right)^{\alpha}\left(\frac{|\Delta|}{|J|}\right)^{\alpha+\delta}\left(1+\frac{|t_I - t_\Delta|}{|I|}\right)^{-\beta}\left(1+\frac{|t_\Delta - t_J|}{|J|}\right)^{-\beta} \\
&\le \left(\frac{|J|}{|I|}\right)^{\alpha}\sum_{|\Delta|<|J|}\left(\frac{|\Delta|}{|J|}\right)^{2\alpha+\delta}\left(1+\frac{|t_I - t_\Delta|}{|I|}\right)^{-\beta}\left(1+\frac{|t_\Delta - t_J|}{|J|}\right)^{-\beta} \\
&\le \left(\frac{|J|}{|I|}\right)^{\alpha}\sum_{j=1}^{\infty}2^{-j(2\alpha+\delta)}\sum_{\Delta\in\mathcal{D}_{\nu+\mu+j}}\left(1+\frac{|t_I - t_\Delta|}{|I|}\right)^{-\beta}\left(1+\frac{|t_\Delta - t_J|}{|J|}\right)^{-\beta}.
\end{aligned}$$

We apply Lemma 8.1 to the last sum (over $\Delta \in \mathcal{D}_{\nu+\mu+j}$) to obtain

$$\sigma_1 \le C\left(\frac{|J|}{|I|}\right)^{\alpha}\left(1+\frac{|t_I - t_J|}{|I|}\right)^{-\beta}\sum_{j=1}^{\infty}2^{-j(2\alpha+\delta-1)} \le C\left(\frac{|J|}{|I|}\right)^{\alpha}\left(1+\frac{|t_I - t_J|}{|I|}\right)^{-\beta}. \tag{8.3}$$

To estimate σ_2 we use again (3.18) and (3.19) and find

$$\begin{aligned}\sigma_2 &\le \sum_{|J|\le|\Delta|\le|I|}\left(\frac{|\Delta|}{|I|}\right)^{\alpha}\left(\frac{|J|}{|\Delta|}\right)^{\alpha+\delta}\left(1+\frac{|t_I-t_\Delta|}{|I|}\right)^{-\beta}\left(1+\frac{|t_\Delta-t_J|}{|\Delta|}\right)^{-\beta}\\ &\le \left(\frac{|J|}{|I|}\right)^{\alpha}\sum_{j=0}^{\mu}2^{-j\delta}\sum_{\Delta\in\mathcal{D}_{\nu+j}}\left(1+\frac{|t_I-t_\Delta|}{|I|}\right)^{-\beta}\left(1+\frac{|t_\Delta-t_J|}{|\Delta|}\right)^{-\beta}.\end{aligned}$$

We apply Lemma 8.1 to the last sum above to find

$$\sigma_2 \le C\left(\frac{|J|}{|I|}\right)^{\alpha}\left(1+\frac{|t_I-t_J|}{|I|}\right)^{-\beta}\sum_{j=0}^{\mu}2^{-j\delta} \le C\left(\frac{|J|}{|I|}\right)^{\alpha}\left(1+\frac{|t_I-t_J|}{|I|}\right)^{-\beta}. \tag{8.4}$$

Finally, we estimate σ_3. Using again (3.18) and (3.19), we obtain

$$\begin{aligned}\sigma_3 &\le \sum_{|\Delta|>|I|}\left(\frac{|I|}{|\Delta|}\right)^{\alpha}\left(\frac{|J|}{|\Delta|}\right)^{\alpha+\delta}\left(1+\frac{|t_I-t_\Delta|}{|\Delta|}\right)^{-\beta}\left(1+\frac{|t_\Delta-t_J|}{|\Delta|}\right)^{-\beta}\\ &\le \left(\frac{|J|}{|I|}\right)^{\alpha}\sum_{j=1}^{\infty}2^{-j(2\alpha+\delta)}\sum_{\Delta\in\mathcal{D}_{\nu-j}}\left(1+\frac{|t_I-t_\Delta|}{|\Delta|}\right)^{-\beta}\left(1+\frac{|t_\Delta-t_J|}{|\Delta|}\right)^{-\beta}.\end{aligned}$$

Applying Lemma 8.1 to the last sum above (over $\Delta\in\mathcal{D}_{\nu-j}$), we find

$$\begin{aligned}\sigma_3 &\le C\left(\frac{|J|}{|I|}\right)^{\alpha}\sum_{j=1}^{\infty}2^{-j(2\alpha+\delta)}\left(1+\frac{|t_I-t_J|}{2^{-\nu+j}}\right)^{-\beta}\\ &\le C\left(\frac{|J|}{|I|}\right)^{\alpha}\left(1+\frac{|t_I-t_J|}{|I|}\right)^{-\beta}\sum_{j=1}^{\infty}2^{-j(2\alpha+\delta-\beta)}\\ &\le C\left(\frac{|J|}{|I|}\right)^{\alpha}\left(1+\frac{|t_I-t_J|}{|I|}\right)^{-\beta},\end{aligned}$$

where we used that $2\alpha\ge\beta$ and

$$1+\frac{|t_I-t_J|}{2^{-\nu+j}} \ge \frac{1}{2^j}\left(1+\frac{|t_I-t_J|}{2^{-\nu}}\right) = \frac{1}{2^j}\left(1+\frac{|t_I-t_J|}{|I|}\right).$$

The above estimates for σ_3, (8.3), and (8.4) imply (3.20) in the case when $|I|\ge|J|$. The proof of (3.20) when $|I|<|J|$ is quite similar and will be omitted. Lemma 3.3 is proved. □

8.2 Proof of Proposition 4.1.

We shall only prove that conditions (i) – (iii) from Proposition 4.1 imply that $\{\theta_\nu\}_{\nu=1}^{\infty}$ is a Schauder basis for X. To this end it is sufficient to prove that each $f\in X$ has the representation

$$f=\sum_{\nu=1}^{\infty}\langle f,\tilde{\theta}_\nu\rangle\theta_\nu \quad \text{in} \quad X. \tag{8.5}$$

The uniqueness of this representation follows by (i) and (ii).

Let $\epsilon > 0$. Since $\{\theta_\nu\}_{\nu=1}^\infty$ is complete in X, there exists

$$f_N = \sum_{\nu=1}^N a_\nu \theta_\nu \quad \text{such that} \quad \|f - f_N\| < \epsilon.$$

We now select m_1 so that $2^{m_1} \geq N$. Let $n \geq 2^{m_1}$. Then there exists $m \geq m_1$ such that $n = 2^m + i$ with $0 \leq i < 2^m$. Since $f_N \in X_m$, then

$$\sum_{\nu=1}^{2^m} \langle f_N, \tilde{\omega}_{\nu,m}\rangle \omega_{\nu,m} = f_N. \tag{8.6}$$

As we pointed out in Remark 4.1, it follows, by (i) – (iii), that $\{\tilde{\theta}_{\nu,m}\}_{\nu=2^m+1}^\infty = \{\tilde{\theta}_\nu\}_{\nu=2^m+1}^\infty$. From this, it readily follows that, for each $g \in X$,

$$\sum_{\nu=1}^{2^m} \langle g, \tilde{\theta}_\nu\rangle \theta_\nu = \sum_{\nu=1}^{2^m} \langle g, \tilde{\omega}_{\nu,m}\rangle \omega_{\nu,m}. \tag{8.7}$$

We use now (4.1), (8.6), and (8.7) to obtain

$$\begin{aligned}
&\|f - \sum_{\nu=1}^n \langle f, \tilde{\theta}_\nu\rangle\theta_\nu\| \leq \|f - f_N\| + \|f_N - \sum_{\nu=1}^n \langle f, \tilde{\theta}_\nu\rangle \theta_\nu\| \\
\leq\ & \epsilon + \|\sum_{\nu=1}^{2^m} \langle f_N, \tilde{\omega}_{\nu,m}\rangle\omega_{\nu,m} - \sum_{\nu=1}^n \langle f, \tilde{\theta}_\nu\rangle\theta_\nu\| \\
=\ & \epsilon + \|\sum_{\nu=1}^{2^m} \langle f_N, \tilde{\omega}_{\nu,m}\rangle\omega_{\nu,m} + \sum_{\nu=2^m+1}^{2^m+i} \langle f_N, \tilde{\theta}_\nu\rangle\theta_\nu \\
&- \sum_{\nu=1}^{2^m} \langle f, \tilde{\omega}_{\nu,m}\rangle\omega_{\nu,m} - \sum_{\nu=2^m+1}^{2^m+i} \langle f, \tilde{\theta}_\nu\rangle\theta_\nu\| \\
=\ & \epsilon + \|\sum_{\nu=1}^{2^m} \langle f_N - f, \tilde{\omega}_{\nu,m}\rangle\omega_{\nu,m} + \sum_{\nu=2^m+1}^{2^m+i} \langle f_N - f, \tilde{\theta}_\nu\rangle\theta_\nu\| \\
\leq\ & \epsilon + K\|f - f_N\| \leq (K+1)\epsilon.
\end{aligned}$$

Therefore (8.5) holds. Proposition 4.1 is proved. □

8.3 Proof of Lemma 5.1.

We first prove (a). Without loss of generality we can assume that $t_J = 0$. Let $\mu \leq \nu$. We denote

$$\mathcal{E}_{J,0} := \{\Delta \in \mathcal{D}_\mu : 2^\mu |t_\Delta| \leq 1\}$$

and

$$\mathcal{E}_{J,j} := \{\Delta \in \mathcal{D}_m : 2^{j-1} < 2^\mu |t_\Delta| \leq 2^j\}, \quad j = 1, 2, \ldots.$$

We have

$$\sum_{\Delta\in\mathcal{E}_{J,j}} |h_\Delta|(1+2^\mu|t_\Delta|)^{-\beta} \le C2^{-j\beta} \sum_{\Delta\in\mathcal{E}_{J,j}} |h_\Delta|$$

$$\le C2^{-j\beta}2^\mu \int_{\mathbb{R}} \sum_{\Delta\in\mathcal{E}_{J,j}} |h_\Delta|\chi_\Delta(x)\,dx$$

$$\le C2^{-j\beta}2^\mu |\cup_{\Delta\in\mathcal{E}_{J,j}}\Delta||\cup_{\Delta\in\mathcal{E}_{J,j}}\Delta|^{-1} \int_{\cup_{\Delta\in\mathcal{E}_{J,j}}\Delta} \sum_{\Delta\in\mathcal{E}_{J,j}} |h_\Delta|\chi_\Delta(x)\,dx$$

$$\le C2^{-j(\beta-1)} M\left(\sum_{\Delta\in\mathcal{D}_\mu} |h_\Delta|\chi_\Delta\right)(t), \quad t\in J.$$

Summing over $j=0,1,\dots$, we obtain, for $t\in J$,

$$\sum_{\Delta\in\mathcal{D}_\mu} |h_\Delta|(1+2^\mu|t_\Delta|)^{-\beta} \le C\sum_{j=0}^{\infty} 2^{-j(\beta-1)} M\left(\sum_{\Delta\in\mathcal{E}_{J,j}} |h_\Delta|\chi_\Delta\right)(t)$$

$$\le C\left(\sum_{j=0}^{\infty} 2^{-j(\beta-1)}\right) M\left(\sum_{\Delta\in\mathcal{D}_\mu} |h_\Delta|\chi_\Delta\right)(t)$$

$$\le CM\left(\sum_{\Delta\in\mathcal{D}_\mu} |h_\Delta|\chi_\Delta\right)(t).$$

Thus (5.10) is proved.

We now prove (5.11). Let us assume again that $t_J=0$. Let $\mu>\nu$. We now denote

$$\mathcal{F}_{J,0} := \{\Delta\in\mathcal{D}_\mu : 2^\nu|t_\Delta|\le 1\}$$

and

$$\mathcal{F}_{J,j} := \{\Delta\in\mathcal{D}_\mu : 2^{j-1} < 2^\nu|t_\Delta|\le 2^j\}, \quad j=1,2,\dots.$$

We find

$$\sum_{\Delta\in\mathcal{F}_{J,j}} |h_\Delta|(1+2^\nu|t_\Delta|)^{-\beta}$$

$$\le C2^{-j\beta} \sum_{\Delta\in\mathcal{F}_{J,j}} |h_\Delta| \le C2^{-j\beta}2^\mu \int_{\mathbb{R}} \sum_{\Delta\in\mathcal{F}_{J,j}} |h_\Delta|\chi_\Delta(x)\,dx$$

$$\le C2^{-j\beta}2^\mu |\cup_{\Delta\in\mathcal{F}_{J,j}}\Delta||\cup_{\Delta\in\mathcal{F}_{J,j}}\Delta|^{-1} \int_{\cup_{\Delta\in\mathcal{F}_{J,j}}\Delta} \sum_{\Delta\in\mathcal{F}_{J,j}} |h_\Delta|\chi_\Delta(x)\,dx$$

$$\le C2^{-j(\beta-1)}2^{\mu-\nu} M\left(\sum_{\Delta\in\mathcal{D}_\mu} |h_\Delta|\chi_\Delta\right)(t), \quad t\in J.$$

Summing over $j = 0, 1, \ldots$, we obtain, for $t \in J$,

$$\sum_{\Delta \in \mathcal{D}_\mu} |h_\Delta|(1 + 2^\nu |t_\Delta|)^{-\beta} \leq C2^{\mu-\nu} \sum_{j=0}^{\infty} 2^{-j(\beta-1)} M \left(\sum_{\Delta \in \mathcal{F}_{J,j}} |h_\Delta| \chi_\Delta \right)(t)$$

$$\leq C2^{\mu-\nu} \left(\sum_{j=0}^{\infty} 2^{-j(\beta-1)} \right) M \left(\sum_{\Delta \in \mathcal{D}_\mu} |h_\Delta| \chi_\Delta \right)(t)$$

$$\leq C2^{\mu-\nu} M \left(\sum_{\Delta \in \mathcal{D}_\mu} |h_\Delta| \chi_\Delta \right)(t).$$

Thus (5.11) is proved. This completes the proof of Lemma 5.1. □

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