

# VARIOUS TOPOLOGIES ON TREES

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ABSTRACT. This is a survey article on trees, with a modest number of proofs to give a flavor of the way these topologies can be efficiently handled. Trees are defined in set-theoretic fashion as partially ordered sets in which the elements below each element are well-ordered. A number of different topologies on trees are treated, some at considerable length. Two sections deal in some depth with the coarse and fine wedge topologies, and the interval topology, respectively. The coarse wedge topology gives a class of supercompact monotone normal topological spaces, and the fine wedge topology puts a monotone normal, hereditarily ultraparacompact topology on every tree. The interval topology gives a large variety of topological properties, some of which depend upon set-theoretic axioms beyond ZFC. Many of the open problems in this area are given in the last section.

## 1. Trees as abstractions

Trees, in the everyday sense of the word, generally have a property that lends itself almost irresistibly to abstraction. This is the property of repeated branching without rejoining: once the trunk branches off, once a branch branches further, etc. there is no subsequent re-combination.

Hence, one hears of such abstractions as “phone trees,” “decision trees,” and “phylogenetic trees,” which are based on this, as well as “family trees” which usually conform to it if they just list the near ancestors of a person or that person’s near descendants. Variants of the Latin word “ramus”, such as “ramified” and “ramification”, are also frequently employed in such abstractions.

The abstraction that will take up most of this article is the one used by most researchers in set theory and allied fields, with one insignificant (but confusing if one is not alert) variant.

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**1.1. Definition.** A *tree* is a partially ordered set in which the predecessors of any element are well-ordered. [Given two elements  $x < y$  of a poset, we say  $x$  is a *predecessor* of  $y$  and  $y$  is a *successor* of  $x$ .]

The insignificant variant is that logicians generally put “successors” in place of “predecessors”, prompting comments that they are really talking about “root systems”.

It follows from Definition 1.1 that each tree has a set of minimal members, above which every member of the tree is to be found. The botanical language continues with:

**1.2. Definition.** If a tree has only one minimal member, it is said to be *rooted* and the minimal member is called the *root* of the tree. Maximal members (if any) of a tree are called *leaves*, and maximal chains are called *branches*.

[Recall that a *chain* in a poset is a totally ordered subset. There is some conflict in the usage of “antichain” where partially ordered sets in general are concerned, but fortunately they coincide for trees: an *antichain* in a tree is a set of pairwise incomparable elements.]

Note the slight deviation from everyday talk: a branch always goes down to the bottom level of the tree. There are standard notations for the levels of any tree; the main versions are the one adopted here and the one that uses subscripts, putting  $T_\alpha$  where we will use  $T(\alpha)$ .

**1.3. Notation.** If  $T$  is a tree, then  $T(0)$  is its set of minimal members. Given an ordinal  $\alpha$ , if  $T(\beta)$  has been defined for all  $\beta < \alpha$ , then  $T \upharpoonright \alpha = \bigcup\{T(\beta) : \beta < \alpha\}$ , while  $T(\alpha)$  is the set of minimal members of  $T \setminus T \upharpoonright \alpha$ . The set  $T(\alpha)$  is called *the  $\alpha$ -th level of  $T$* .

We use the usual notation for intervals, such as  $[s, t) = \{x \in T : s \leq x < t\}$ , and we also adopt the following suggestive notation.

**1.4. Notation.** Given elements  $s < t$  of a tree  $T$ , let  $V_t = \{s \in T : s \geq t\}$ , and we let  $\hat{t} = \{s \in T : s \leq t\}$ , given  $A \subset T$ , let  $V_A = \bigcup\{V_a : a \in A\}$  and let  $\hat{A} = \bigcup\{\hat{a} : a \in A\}$ .

**1.5. Definition.** The *height* of a tree  $T$  is the least ordinal  $\alpha$  such that  $T(\alpha) = \emptyset$ . Given a cardinal  $\kappa$  and an ordinal  $\alpha$ , the *full  $\kappa$ -ary tree of height  $\alpha$*  is the tree of all transfinite sequences  $f : \beta \rightarrow \kappa$ , for some ordinal  $\beta < \alpha$ , and the order on the tree is end extension:  $f \leq g$  iff  $\text{dom } f \subset \text{dom } g$  and  $g \upharpoonright \text{dom } f = f$ .

The numberings involved in this definition are a bit tricky. The full binary tree of height  $\omega$  has no elements at level  $\omega$  and all its elements are finite sequences of 0’s and 1’s. The full binary tree of height  $\omega + 1$ , also known as the Cantor tree, has members on its top level which are ordinary sequences: there is no  $\omega$ -th term, let alone an  $\omega + 1$ -st term. There are trees of height  $\omega_1$  with no branches of length  $\omega_1$ ,

such as the tree of ascending sequences of real numbers, ordered by end extension. There are also easy examples of trees of height  $\omega$  with no infinite branches.

In drawing diagrams of trees, it is traditional to draw line segments joining elements to their immediate successors. These lines are usually not meant to be parts of the trees; if they are, point-set topologists usually call the resulting objects “road spaces” [see Figure 1]. For example, what Steen and Seebach [26] refer to as the “Cantor tree” is more usually called the “Cantor road space,” and is formed from the Cantor tree by this process of joining successive elements with unit intervals. The Moore road space, which Steen and Seebach mention but do not define explicitly, can be formed from the Cantor road space by adding copies of the unit interval as successors to each point on the  $\omega$ -th level of the Cantor tree.

[INSERT FIG. 1 HERE OR CLOSE TO HERE.]

In other branches of topology, and elsewhere in mathematics and the sciences, it may be a different story. One popular definition among topologists (cf. [19]) is “simply connected graph,” where a graph is defined as a nonempty connected 1-complex. For such spaces, “simply connected” is equivalent to any two distinct points being the endpoints of a unique arc.

Biologists use “trees” in a similar way. Some biologists use the line segments in their phylogenetic trees to represent the actual species studied, while each fork in their trees represents a speciation event. Topologically, they treat their trees as though they were subsets of the plane, and they quite correctly observe that the topology depends on the actual evolutionary events. Indeed, once the root of the tree is identified, two topologies are equal if and only if they depict the same phylogenetic relationship of the species defined.

To minimize confusion between this kind of tree and the trees of Definition 1.1, I will only use the word “tree” in this other way one more time, at the end of the following section.

## 2. Comparison of topologies and their convergence properties

There are many topologies which flow naturally out of the order structure of trees. The ones we discuss have fairly straightforward generalizations to partially ordered sets, and some have already been applied to more general kinds of posets.

Examples 1 through 4 below will receive additional attention in later sections and so we will concentrate on aspects of their convergence in this section. We will only need a few concepts pertaining to convergence since the topologies we consider are all quite well behaved.

**2.1. Definition** A space  $X$  is *Fréchet-Urysohn* [resp. *radial*] if, whenever a point  $x$  is in the closure of  $A \subset X$ , there is a sequence [resp. a well-ordered net] in  $A$  converging to  $x$ .

A pair of more general concepts will be defined below (Definition 2.3.).

We begin with the topology that is often simply called “the tree topology” by set-theoretic topologists.

**Example 1.** *The interval topology on a tree  $T$  is the one whose base is all sets of the form  $(s, t] = \{x \in T : s < x \leq t\}$ , together with all singletons  $\{t\}$  such that  $t$  is a minimal member of  $T$ .*

It is easy to see that every tree is radial in the interval topology, and that a tree is Fréchet-Urysohn in this topology iff it is of height  $\leq \omega_1$  iff it is first countable. In fact, a point  $t$  is an accumulation point of a set  $A$  if, and only if, it is in the closure of  $A \cap \hat{t}$ , and we can order the elements of this set in their natural order to produce a well-ordered net converging to  $t$ , and if it is of countable cofinality then we have a sequence converging to  $t$ .

Every tree is locally compact in the interval topology, and is Hausdorff (hence Tychonoff, and 0-dimensional) iff its pseudo-suprema of nonempty chains all consist of one point:

**2.2 Definition.** Given a chain  $C$  in a tree  $T$  such that  $C$  is bounded above, the *pseudo-supremum* of  $C$  is the set of minimal upper bounds of  $C$ ; in other words, the set of minimal members of  $\{t \in T : c \leq t \text{ for all } c \in C\}$ .

Pseudo-suprema are always nonempty because they are only defined for chains that are bounded above. By the usual conventions, the pseudo-supremum of the empty chain is the bottom level  $T(0)$  of  $T$ . This is called the trivial pseudo-supremum, and all pseudo-suprema of nonempty chains will be called non-trivial.

**Example 2.** *The fine wedge topology on a tree is the topology whose subbase is the collection of all sets  $V_t$  and their complements.*

It is easy to see that a local base at each point  $t$  is formed by sets of the form

$$W_t^F = V_t \setminus \bigcup \{V_s : s \in F\} = V_t \setminus V_F$$

where  $F$  is a finite set of successors of  $t$ . Of course, we can restrict ourselves to immediate successors for membership in  $F$ .

It follows from this that a point is isolated in the fine wedge topology iff it has at most finitely many immediate successors, and is a point of first countability iff it has at most countably many immediate successors. But in any case, the topology is always Fréchet-Urysohn. Indeed,  $t$  is in the closure of  $A$  iff  $A$  meets  $V_s$  for infinitely many immediate successors  $s$  of  $t$ , and a sequence  $\langle a_n : n \in \omega \rangle$  in  $A$  will converge to  $t$  iff only finitely many  $a_n$  are above the same successor of  $t$  and only finitely many are outside  $V_t$ .

The name for the following topology is inspired by the shape of its basic open sets.

**Example 3.** *The **chevron topology** on a tree  $T$  is the one whose base consists of all  $\{m\}$  such that  $m$  is minimal in  $T$ , together with all sets of the form*

$$C[s, t] = (V_s \setminus V_t) \cup \{t\}$$

*such that  $s \leq t$ , where  $s$  is either minimal or on a successor level.*

Every tree is radial in the chevron topology. Indeed, a point  $t$  is in the closure of  $A$  if, and only if, either  $t \in A$  or  $t$  is on a limit level and  $A$  meets  $V_x \setminus V_t$  for cofinally many  $x \in \hat{t} \setminus \{t\}$ . In the latter case, we can select, for each  $x < t$ , an element  $a_x \in A$  such that  $a_x \in V_y \setminus V_t$  for some  $y \in [x, t)$  and then the well-ordered net  $\langle a_x : x < t \rangle$  converges to  $t$ .

It is easy to see that the characters of points are the same in the interval and chevron topologies; in particular, the same points are isolated in both topologies. Of course, the chevron topology is coarser than the interval topology, and strictly coarser in many trees, such as Cantor tree, where the top level is easily seen to be homeomorphic to the Cantor set in the chevron topology. In the lattice of all topologies on  $T$ , the least upper bound [called the *join*] of the chevron topology with the fine wedge topology is the discrete topology since  $t$  is the only point in  $C[s, t] \cap V_t$ .

The next four topologies all coincide for trees in which nontrivial pseudo-suprema are suprema. They also agree on the relative topology which results when all nontrivial pseudo-suprema of more than one point are removed from the tree. [This should not be confused with the topology on the resulting tree that satisfies the formal definition of the respective examples.] Example 4b is the coarsest possible topology which produces such agreement, while Example 4c is the finest.

**Example 4a.** *The **split wedge topology** is the greatest lower bound [i.e., the meet] of the chevron and fine wedge topologies in the lattice of topologies on  $T$ .*

Note that in a finitary tree [that is, one in which no element has infinitely many immediate successors] the split wedge topology and chevron topology coincide: the tree is simply discrete in the fine wedge topology.

In other trees, we can construct local bases at each point in the split wedge topology by letting their members be simply the union of a basic chevron neighborhood and a basic fine wedge neighborhood. Indeed, the resulting set is open in both topologies, hence in their meet. Because of this, every tree is radial in the split wedge topology: given  $A$  with  $t$  in its closure,  $t$  must be in the closure either of  $A \setminus V_t$  or of  $A \cap V_t$ ; and then we follow the argument for the corresponding finer topology.

The following topology differs from the split wedge topology only in that sets of pseudo-suprema [except for the trivial pseudo-supremum  $T(0)$ ] are indiscrete rather than discrete in the relative topology. This allows a third possibility for points to be in the closure of  $A$ , but every constant net in an indiscrete space converges to every point in the space, so the following topology is again radial.

**Example 4b.** *The coarse wedge topology on a tree  $T$  is the one whose subbase is the set of all wedges  $V_t$  and their complements, where  $t$  is either minimal or on a successor level.*

If  $t$  is minimal or on a successor level, then a local base is formed by the sets  $W_t^F$  exactly as in the fine wedge topology, with  $F$  a finite set of immediate successors of  $t$ . If, on the other hand,  $t$  is on a limit level, then one must use  $W_s^F$  such that  $s$  is on a successor level below  $t$ . However, the most appropriate  $F$  to take are not sets of immediate successors of  $s$  but sets of immediate successors of  $t$ . Given any  $W_s^F$  containing  $t$ , one can find  $t' \in [s, t)$  so that the only members of  $F$  above  $t'$  are those above  $t$ , and then  $W_{t'}^G$  is of this form, where  $G = F \cap V_t$ .

An attractive feature of the coarse wedge topology is that it always has a base of clopen sets, even if some nontrivial pseudo-suprema are not suprema. The fine wedge topology and the next example are the only other ones that have this feature, of the topologies considered here.

**Example 4c.** *The Lawson topology on a tree  $T$  is the one whose subbase is the set of all wedges  $V_t$  and their complements, where  $t$  is not the supremum of a nonempty chain in  $\hat{t} \setminus \{t\}$ .*

The Lawson topology and the fine wedge topologies are the only ones which are invariably Hausdorff for all trees. The Lawson topology is radial, by the same argument as for the split wedge topology. Of course, points in pseudo-suprema with more than one element are more easily handled in the Lawson topology, because they are isolated.

Another closely related topology is intermediate between the split wedge and coarse wedge topologies, giving sets of pseudo-suprema the cofinite topology. Since every injective  $\omega$ -sequence converges to each point of a space with cofinite topology, this topology too is radial:

**Example 4d.** *The hybrid wedge topology on a tree  $T$  is the one whose subbase consists of all complements of wedges  $V_t$  together with those wedges  $V_s$  for which  $s$  is either minimal or a successor.*

Note that doing it the other way around—all wedges plus complements of wedges based on successors or minimal members—simply produces the fine wedge topology because the basic sets  $W_t^F$  such that  $F$  consists of immediate successors of  $t$ , are all there.

So far, the topologies we have been considering are all Hausdorff and zero-dimensional if all nontrivial pseudo-suprema are singletons, hence suprema (and we can drop the conditional clause for the fine wedge and Lawson topologies). With one exception (Example 7) this is not the case with the remaining topologies of this section. These remaining topologies will not be dealt with in subsequent sections and the reader may skip to Section 3 now or later with no loss of continuity.

The next two topologies are Hausdorff iff they are  $T_1$  iff no element is above any other element, i.e. if  $T(0)$  is all of  $T$ ; and in this case, they are discrete. The first one can be thought of as ‘one half of the Lawson topology’:

**Example 5.** *The Scott topology on a tree  $T$  is the one for which sets of the form  $V_t$  are a base, where  $t$  is not the supremum of a nonempty chain in  $\hat{T} \setminus \{t\}$ .*

For arbitrary posets, one has to use a different description, easily seen equivalent for trees: the Scott-open subsets of a poset  $P$  are those upper sets  $U$  such that no member is the supremum of a directed subset of  $P \setminus U$ . [A subset  $U$  of a poset is said to be an *upper set* if  $V_p \subset U$  whenever  $p \in U$ . Thus, for example, the upper subsets of  $\mathbb{R}$  are the right rays, and those of the form  $(a, +\infty)$  are the Scott open subsets.]

Of course, every tree is a  $T_0$ -space in the Scott topology. It is a radial space by the same arguments that apply to the Lawson topology, only they are simpler since any constant sequence in  $V_t$  converges to  $t$  in the Scott topology. This applies *a fortiori* to the next topology, where a point  $t$  is in the closure of  $A$  iff  $A$  meets  $V_t$ .

**Example 6.** *The Alexandroff discrete topology is the one for which all sets of the form  $V_t$  form a base.*

Examples 5 and 6 have the property that the order can be recovered from the topology by setting  $x \leq y$  iff  $x$  is in the closure of  $\{y\}$ .

While these last two topologies may be “uninteresting” from the point of view of most general topologists, they have great significance from other points of view. The Alexandroff discrete topology, generalized to posets, is the one behind words like “open” and “dense” in the applications of forcing.

Forcing is a method of producing models of set theory, pioneered by Paul Cohen, who used it in 1963 to show that the continuum hypothesis is independent of the usual axioms of set theory. It has revolutionized set theory and a number of other branches of mathematics, especially set-theoretic topology and the theory of Boolean algebras.

The Scott topology is important in theoretical computer science (cf. [18]). Appropriately enough, it is named after the leading computer scientist Dana Scott, who showed [25] that continuous lattices equipped with this topology are precisely the injective objects in the category of  $T_0$ -spaces and continuous functions.

When the discrete topology and one more topology are added, and we restrict our attention to trees in which pseudo-suprema are suprema, the foregoing topologies form a sublattice of the lattice of all topologies, as shown in Fig. 2. The pentagon on the right shows that this is not a modular lattice.

[INSERT FIG. 2 HERE OR CLOSE TO HERE.]

I have given the name “fantail topology” to the topology that is the meet of the interval and fine wedge topologies, because of the pictures I associate with the basic neighborhoods as defined below.

**Example 7.** *The fantail topology is the one for which a base is the collection of all sets of the form  $\bigcup\{W_x^{F(x)} : s \leq x \leq t\}$  such that  $s$  is either minimal or a successor, and  $F(x)$  is a finite subset of  $V_x$  for all  $x$ , satisfying the following condition: if  $x < t$  then  $x' \in F(x)$ , where  $x'$  is the immediate successor of  $x$  which satisfies  $x' \leq t$ .*

This is a more complicated topology than the ones considered so far, and it is the only one which is not always radial. However, it is the next best thing in a sense:

**2.3. Definition** A space  $X$  is *pseudo-radial* if closures can be taken by iterating the operation of taking limits of convergent well-ordered nets; the number of iterations required is the *chain-net order* of  $X$ .

More formally: given a set  $A \subset X$ , let  $A^{-1}$  be the set of all points which are limits of well-ordered nets from  $A$ . If  $\alpha$  is an ordinal and  $A^{-\alpha}$  has been defined, let  $A^{-\alpha+1} = (A^{-\alpha})^{-1}$ , while if  $\alpha$  is a limit ordinal we define  $A^{-\alpha}$  to be the union of all the  $A^{-\beta}$  such that  $\beta < \alpha$ . The first ordinal  $\alpha$  such that  $A^{-\alpha+1} = A^{-\alpha}$  for all  $A \subset X$  is the chain-net order of  $X$ , provided  $A^{-\alpha+1} = A^{-\alpha}$  is always the closure of  $A$ ; this condition characterizes pseudo-radial spaces.

**2.4. Theorem.** *Every tree is pseudo-radial, of chain net order  $\leq 2$  in the fantail topology. The order is exactly 2 in any tree of height  $> \omega$  in which every element of  $T(n)$  has infinitely many immediate successors for  $n \in \omega$ .*

Before showing this, it is helpful to make some observations and to define another concept pertaining to general spaces.

We were ‘fortunate’ to have the union of a basic chevron neighborhood and a basic fine wedge neighborhood be open in both topologies. In the case of more general spaces, and in particular the case where the interval topology is substituted for the fantail topology, we can expect only that such unions form a weak base:

**2.5. Definition.** Let  $X$  be a set. A *weak base* on  $X$  is a family of filterbases  $\mathbb{B} = \{\mathcal{B}(x) : x \in X\}$  such that  $x \in B$  for all  $B \in \mathcal{B}(x)$ . The topology induced on  $X$  by  $\mathbb{B}$  is the one in which a set  $U$  is open if, and only if, there exists for each point  $x \in U$  a member  $B$  of  $\mathcal{B}(x)$  such that  $B \subset U$ .

Of course, every system of ordinary neighborhood bases is a weak base, and this is something worth keeping in mind when reading the following lemma.

**2.6. Lemma.** *If  $\tau_1$  and  $\tau_2$  are topologies on  $X$  and  $\mathbb{B}_1$  and  $\mathbb{B}_2$  are weak bases for  $\tau_1$  and  $\tau_2$  respectively, then the weak base  $\mathbb{B}$  in which  $\mathcal{B}(x) = \{B_1 \cup B_2 : B_i \in \mathcal{B}_i(x) \text{ for } i = 1, 2\}$  is a weak base for the meet of the  $\tau_i$ .*

*Proof.* Let  $U$  be open in both  $\tau_i$  — equivalently, in their meet. For each  $x \in U$  and each  $i$  there exists  $B_i \in \mathcal{B}_i(x)$  such that  $B_i(x) \subset U$ , so  $B_1(x) \cup B_2(x) \subset U$ . Conversely, suppose that  $V \subset X$  and for each  $x \in V$ , there exist  $B_i \in \mathcal{B}_i(x)$  such that  $B_1 \cup B_2 \subset V$ . Then  $V$  is open in  $\tau_i$  for both  $i$ , hence in their meet.  $\square$

Now, in the case where the  $\tau_i$  are the fine wedge and interval topologies, letting  $\mathcal{B}_1(x)$  be the local base of all sets  $W_x^F$ , and letting  $\mathcal{B}_2(x)$  be the set of all intervals  $(s, x]$ , gives us a weak base for the fantail topology in which  $B_1 \cup B_2$  is not always open in the fine wedge topology, hence it is not always open in the fantail topology. However, by attaching a fine wedge neighborhood to each point of  $B_2 = (s, t]$  we do produce a set that is open in both topologies, and it is easy to see that every set that is open in both topologies must contain a set of this form; among these are the basic open sets described in the statement of Example 7. They are also easily seen to be closed in both topologies.

*Proof of Theorem 2.4.* First we show that every point in the closure of  $A$  is the limit of a convergent well-ordered net in  $\tilde{A}$  for all  $A \subset T$ , where  $\tilde{A}$  is the set of all limits of convergent sequences in  $A$ . This we do by showing that if  $t$  is in the closure of  $A$ , then either  $t \in \tilde{A}$  or a cofinal subset of  $\hat{t} \setminus \{t\}$  is in  $\tilde{A}$ . Of course, there will be a well-ordered net in this cofinal subset converging to  $t$ .

If  $t$  is in the closure of  $A \cap V_t$ , then there is a sequence in  $A \cap V_t$  converging to  $t$  just as in the fine wedge topology. So suppose not; then  $t$  is in the closure of  $A \setminus V_t$ ; this of course implies that  $t$  is on a limit level. If there were no cofinal subset of  $\hat{t} \setminus \{t\}$  in  $\tilde{A}$ , then we could attach fine wedge neighborhoods missing  $A$  to all members of a final segment  $[s, t]$ , thereby keeping  $t$  out of the closure of  $A \setminus V_t$ ; and thus we can build a basic neighborhood as in the initial presentation of Example 7, missing  $A$ , contradicting the claim that  $t$  is in the closure of  $A$ .

To show that chain-net order is exactly 2 in trees as described in the second sentence, we will produce a copy of the Arens space  $S_2$  in any such tree.

The Arens space can be defined as the space whose underlying set  $S_2$  is  $\{x_\omega\} \cup \{x_n : n \in \omega\} \cup \{x_k^n : n, k \in \omega\}$  faithfully indexed, e.g.  $x_n^k = x_m^i$  iff  $n = m$  and  $k = i$ ; and in which a weak base is given by  $\{\mathcal{B}(p) : p \in S_2\}$  where  $\mathcal{B}(x_k^n) = \{\{x_k^n\}\}$ ,  $\mathcal{B}(x_n)$  is the collection of all sets  $\{x_n\} \cup \{x_k^n : k \geq j\}$  as  $j$  ranges over  $\omega$ ; and  $\mathcal{B}(x_\omega)$  is the collection of all sets  $\{x_\omega\} \cup \{x_j : j \geq n\}$  as  $n$  ranges over  $\omega$ . Some elementary properties of  $S_2$  are that  $\langle x_n : n \in \omega \rangle$  converges to  $x_\omega$ ; that all the points  $x_k^n$  are isolated; and that the points  $x_n (n \in \omega)$  are points of first countability with each set  $\{x_n\} \cup \{x_k^n : k \geq j\}$  a clopen copy of  $\omega + 1$ .

The most crucial feature, for our purposes, is that the set of isolated points has  $x_\omega$  in its closure, yet no sequence of isolated points can converge to  $x_\omega$ . To see this, note that any sequence that meets some ‘column’  $\{x_k^n : k \in \omega\}$  in an infinite set cannot converge to  $x_\omega$ , and also that any sequence of isolated points that meets each ‘column’ in a finite set does not have  $x_\omega$  in its closure, because we can take a member  $B$  of  $\mathcal{B}(x_\omega)$  and attach a clopen set  $\{x_n\} \cup \{x_k^n : k \geq j_n\}$  missing the sequence to each  $x_n \in B$ , producing an open neighborhood of  $x_\omega$  that misses the sequence.

Now it is routine to build a copy of  $S_2$  in any tree as described. Let  $\{x_\alpha : \leq \omega\}$  be represented by  $\hat{t}$  where  $t$  is any point on level  $\omega$ ; of course,  $x_\omega$  is represented by  $t$  itself. For each  $x_n (n \in \omega)$  the sequence  $\{x_n\} \cup \{x_k^n : k \in \omega\}$  is represented by some countably infinite set of immediate successors of  $x_n$  not in  $\hat{t}$ . The weak base given in the paragraph following the proof of Lemma 2.6 traces a weak base on this subspace exactly as in  $S_2$ . Moreover, the resulting copy of  $S_2$  has no more points of the tree in its closure except perhaps points in the pseudo-supremum of  $\{x_n : n \in \omega\}$ ; and these do not alter the fact that  $t$  cannot have any sequence converge to it from the copy of  $S_2$  other than those in which a cofinite subsequence is in  $\{x_n : n \in \omega\}$ .  $\square$

This concludes our treatment of the fantail topology. The next three examples are taken from [18]. Our list of tree topologies in this section will be concluded with their meets and joins with each other and with the topologies given so far. Two of the topologies are an instance of a general motif: given a topology  $\tau$  defined on a class of posets, one obtains the reverse topology  $\tau^d$  on a poset by turning the poset upside down (i.e. reversing the order relation), defining  $\tau$  on the resulting poset, and then turning it back right side up again.

**Example 8.** *The **Scott<sup>d</sup> topology** on a tree  $T$  is the one for which the subsets  $\hat{t}$  form a base for the topology.*

Indeed, if one inverts a tree, no element is the directed supremum of now-lower elements, and so every now-upper set is open. The Scott<sup>d</sup>-topology is obviously coarser than the discrete and interval topologies and is incomparable to all the remaining topologies. Its join with the Alexandroff discrete topology (and hence with all finer topologies) is obviously the discrete topology, and its join with the Scott topology (and hence all others not above the Alexandroff discrete topology) is clearly the interval topology.

The Scott<sup>d</sup>-topology is obviously first countable; in fact,  $\{\hat{t}\}$  is a one-member local base at  $t$ .

**Example 9.** *The **weak topology** on a tree  $T$  is the one in which the sets  $\hat{t}$  form a subbase for the closed sets.*

Since the sets  $\hat{t}$  are downwards closed and linearly ordered, it easily follows that any closed set other than the whole space is a finite union of sets of the form  $\hat{t}$ .

Moreover, a point  $t$  is in the closure of a finite set  $F$  iff it is in  $\hat{F}$  iff it is in  $\hat{s}$  for some  $s \in F$  iff  $V_t$  meets  $F$ . A subset  $A$  is dense in  $T$  iff it either contains an unbounded chain, or the set of suprema of chains in  $A$  contains an infinite antichain. Hence this topology is radial, and is clearly coarser than the Scott topology and hence coarser than all the topologies considered so far except the coarse wedge and Scott<sup>d</sup> topologies. And its dual is weaker than the Scott<sup>d</sup> topology, of course:

**Example 10.** *The weak<sup>d</sup> topology on a tree  $T$  is the one in which the sets  $V_t$  form a subbase for the closed sets.*

In every tree in which pseudo-suprema are suprema, one can just as easily use only those  $V_t$  in which  $t$  is a minimal or successor element. In any event, it is routine to show that a set is closed in the weak<sup>d</sup> topology iff it equals  $V_A$  for an antichain  $A$  such that  $\hat{A}$  is a finitary tree with finitely many minimal members.

This topology is Fréchet-Urysohn, with a local base at  $t$  consisting of sets  $T \setminus V_S$  where  $S$  is a finite set of points of  $T$  that are outside of  $\hat{t}$ , but have all their predecessors in  $\hat{t}$ . This includes the points of  $T(0) \setminus \hat{t}$  by the usual conventions on the empty set.

The join of the weak<sup>d</sup> topology and the Scott topology is the Lawson topology; in fact this is the way the Lawson topology is defined by Lawson in [18]. The join of the weak and weak<sup>d</sup> topology is strictly coarser than the Lawson topology in general, and it is not hard to see that any closed set is a union of a weak<sup>d</sup>-closed set [see description above] together with finitely many intervals of the form  $[s, t]$ .

The meet of the Alexandroff discrete and Scott<sup>d</sup> topologies is the indiscrete topology on every rooted tree. Of course, this also applies when a coarser topology replaces either or both of these, but there are differences in other trees. In case of the Alexandroff discrete and Scott<sup>d</sup> topologies, we simply have the topological direct sum of the indiscrete rooted trees involved. This is also true if the weak<sup>d</sup> topology replaces the Scott<sup>d</sup> topology and/or the Scott topology replaces the Alexandroff discrete topology. On the other hand, the weak topology gives the cofinite topology on those trees which consist of the single level  $T(0)$ . Of course, only Examples 8 and 10 are not finer than the weak topology, and it is a simple matter to see that in both cases, a set is closed in the meet topology if and only if it contains each  $V_t$  that it meets and is either the entire tree, or else it meets  $V_t$  for only finitely many minimal  $t$ .

The remaining meets are more interesting. They are found in the third quadrant of Figure 2, so to speak.

**Example 11.** The meet of the Scott<sup>d</sup>-topology and the chevron topology can be understood via Lemma 2.6 in the same way that the fantail topology is. A weak base at each point  $t$  on a limit level consists of sets of the form  $C[s, t] \cup \hat{t}$  where

$C[s, t]$  is a chevron, while a weak base at a point  $s$  on a successor or minimum level consists simply of  $\{\hat{s}\}$ . Construction of a base can be worked out as for the fantail topology. I picture a typical member informally as a feather with a wedge cut in the top and finitely many indentations going all the way to the central shaft, each indentation going a finite number of steps up the central shaft. More formally: a local base at  $t$  consists of sets of the form  $\hat{t} \cup (V_{t_0} \setminus \bigcup\{V_x : x \in A\})$  where  $t_0$  is the minimal element of  $\hat{t}$ , and  $A$  is a finite union of levels of  $T$  including the one on which  $t$  itself is to be found. One cannot exclude infinitely many such levels without causing trouble at the next limit level. This fact makes it routine to show that any tree is Fréchet-Urysohn in this topology.

**Example 12.** Similarly, the meet of the Scott<sup>d</sup>-topology with the fine wedge topology (and hence with the fantail topology) can be characterized as the one whose weak base at  $t$  is formed by attaching sets of the form  $W_t^F$  to  $\hat{t}$ . In forming the base, one could just follow the description of the fantail topology, just making sure that the basic neighborhoods start with the minimum point  $t_0$  of  $\hat{t}$ . Like the fantail topology, this one is pseudo-radial of order  $\leq 2$ .

**Example 13.** Examples 11 and 12 are incomparable; their meet has a base formed by taking a basic Example 11 neighborhood and attaching a set  $W_x^F$  to each of the finitely many points of  $\hat{t}$  in the levels met by  $A$ . Of course this is also the meet of the Scott<sup>d</sup>-topology and the split wedge topology, so it is finer than the weak<sup>d</sup>-topology.

Meets involving the coarse wedge, hybrid wedge, and Lawson topologies are left as an exercise for the interested reader.

We close this section with some comments about the Scott topology on phylogenetic trees. This seems to be the appropriate topology for the branch of systematics called cladistics, which is centered on those groups of organisms which form clades. Clades are simply sets of organisms represented by the various  $V_t$  in a phylogenetic tree, and many cladists will refuse to even consider taxa that are not clades as legitimate scientific entities. Their rationale [which I consider to be inadequate] is that one can recover the entire order on the tree by just knowing what the clades are. This is, of course, a very useful thing to be able to do, and is very closely related to the fact that the Scott topology allows us to recover the order on the tree.

### 3. Completeness and Compactness

In this section we consider some elementary aspects of completeness of a tree which depend only on the order, and the kinds of compactness they give rise to.

Where trees are concerned, the very fundamental concept of Dedekind completeness simply translates to every pseudo-supremum being a supremum. Hence it is easy to produce a Dedekind completion for any tree: just give every set of pseudo-suprema of more than one element an immediate predecessor. Note, however, that

the inclusion map of the original tree in its Dedekind completion is not an embedding in many of the topologies of Section 2 [in particular, not in Examples 1 through 4b nor for 4d; the Lawson topology is a noteworthy exception]. This is because the points of the original pseudo-suprema are no longer in the closure of the points on the earlier levels in most of the topologies. On the other hand, the map does have dense range in most of the topologies, including the Lawson topology. Of course, it is always an order-embedding.

Being Dedekind complete is equivalent to the tree being Hausdorff (also to being Tychonoff) in the coarse wedge topology and all finer topologies considered in Section 2. It is also equivalent to the space having a base of clopen sets in the Lawson, chevron, hybrid, and interval topologies, inasmuch as these are always  $T_1$ . Since this is not the sort of property one usually associates with Dedekind completeness, I am being quite sparing of the term in this article where trees are concerned. However, in one case it does seem appropriate:

**3.1 Theorem.** *Let  $T$  be a tree. The following are equivalent.*

- (i)  *$T$  is rooted and Dedekind complete.*
- (ii)  *$T$  is a semilattice downwards; that is, any two elements have a greatest lower bound.*
- (iii) *Every nonempty subset of  $T$  has a greatest lower bound.*

*Proof.* (i) implies (iii): Let  $A$  be a nonempty subset of  $T$  and let  $B$  be the set of lower bounds for  $A$ ;  $B$  is nonempty since the tree is rooted and it is clearly a chain. Since the levels of the tree are well-ordered,  $B$  has a pseudo-supremum on the first level above which there are no members of  $B$ , and since it is a supremum of  $B$  it is also the greatest lower bound of  $A$ .

(iii) implies (ii): Obvious.

(ii) implies (i): It is obvious that  $T$  cannot have more than one minimal element. If  $T$  had a pseudo-supremum that is not a supremum, then any two distinct elements of this pseudo-supremum would fail to have a greatest lower bound.  $\square$

Actually, the equivalence of (ii) and (iii) is part of a more general phenomenon: every infimum is the infimum of some two-element subset. Formally:

**3.2 Theorem.** *Let  $A$  be a set of two or more elements of a tree  $T$  such that  $A$  has a greatest lower bound. Then there are elements  $a_1$  and  $a_2$  of  $A$  such that the g.l.b. of  $a_1$  and  $a_2$  is the g.l.b. of  $A$ .*

*Proof.* Let  $t$  be the g.l.b. of  $A$ . There are at least two distinct immediate successors of  $t$  with elements of  $A$  above them, and we choose  $a_1$  and  $a_2$  from above two of them.  $\square$

Even where there is no greatest lower bound, one can speak of “pseudo-infima” in analogy with pseudo-suprema. Then every nonempty subset of every tree has a

pseudo-infimum, and if  $A$  has at least two elements, we can find two whose pseudo-infimum is the pseudo-infimum of  $A$ .

The use of “complete” does seem quite appropriate in the following concepts, and leads to some nice compactness results.

**3.3 Definition** A tree is *branch-complete* if every branch has a greatest element. A tree is *chain-complete* if every chain has a supremum.

Branch-completions can trivially be produced by giving any branch a greatest element if it does not already have one. The original tree is densely embedded in the resulting tree in Examples 1 through 4d, except for the fine wedge topology. Chain-completions can simply be produced by taking a Dedekind completion of a branch completion, or vice versa; if the descriptions given above are followed, the same tree results no matter which completion is taken first.

The following theorem identifies a rich source of well-behaved examples of compact Hausdorff spaces.

**3.4. Theorem.** *Let  $T$  be a tree. The following are equivalent.*

- (i)  $T$  is branch-complete and has finitely many minimal elements.
- (ii)  $T$  is compact in the coarse wedge topology.

*Proof.* (i) implies (ii): We will show, in fact, that if  $T$  is rooted, it is *supercompact*; that is, it has a subbase such that every open cover has a subcover by two or fewer members. This implies compactness by Alexander’s subbase theorem. The result for non-rooted trees follows immediately since they are topological direct sums of rooted ones in the coarse wedge topology.

Let  $\{V_x : x \in A\}$  and  $\{T \setminus V_x : x \in B\}$  be a subbasic open cover of  $T$ . If  $A \cap B \neq \emptyset$ , then we immediately have a two-member subcover. Otherwise, pick a member of the cover containing the root  $t_0$  of the tree. If this is of the form  $V_x$  we are done since  $x = t_0$  and  $V_x$  is all of  $T$ . Otherwise, every member of the cover containing  $t_0$  is of the form  $T \setminus V_x$ . If  $B$  has a pair of incomparable elements, say  $x$  and  $y$ , then  $\{T \setminus V_x, T \setminus V_y\}$  is a subcover.

It remains to consider the case where  $B$  is a chain. By branch-completeness,  $B$  has a pseudo-supremum  $P$ . The points of  $P$  can only be covered by a set of the form  $V_a$  with  $a < p$  for all  $p \in P$ . But then there exists  $b \in B$  such that  $b > a$ , and then  $\{V_a, T \setminus V_b\}$  is as desired.

(ii) implies (i) Since the minimal level of a tree is closed discrete in the coarse wedge topology,  $T$  can have only finitely many elements in this level if it is to be even countably compact. Also every branch must have a maximum member, for if  $B$  violates this,  $\{T \setminus V_b : b \in B, b \text{ is a successor or minimal}\}$  is an open cover with no finite subcover.  $\square$

**3.5. Corollary.** *A tree is compact Hausdorff in the coarse wedge topology if, and only if, it is chain-complete and has finitely many minimal elements.  $\square$*

The proof of the following theorem will appear in a forthcoming paper. Note the absence of any completeness condition.

**3.6. Theorem.** *Let  $T$  be a tree. The following are equivalent.*

- (1)  *$T$  has countably many minimal elements.*
- (2)  *$T$  is  $\omega_1$ -compact in the coarse wedge topology; that is, every closed discrete subspace is countable.  $\square$*

The foregoing results remain true if the hybrid wedge topology replaces the coarse wedge topology, except that the simple proof of supercompactness in Theorem 3.4 may fail even if the tree is rooted. The proof of the second implication goes through with no change except for the name of the topology. For the first implication, we make a minor modification if  $P$  has finitely many elements; in that case, there is the additional possibility that finitely many  $V_p$  round out the subcover.

The split wedge topology does not add any new compact examples since it coincides with the hybrid wedge topology when pseudo-suprema are finite, and has an infinite closed discrete subspace otherwise. Of course, this applies also to the Lawson topology. A similar statement holds for the chevron topology: it is compact iff it coincides with the hybrid wedge topology and the latter is compact. Equivalently, the tree is finitary and every chain has a finite pseudo-supremum.

Theorem 3.4 and Corollary 3.5 have straightforward analogues for countably compact spaces. Proofs are left as an exercise for the reader:

**3.7. Theorem.** *Let  $T$  be a tree with the coarse wedge or hybrid wedge topology. The following are equivalent.*

- (i)  *$T$  has finitely many minimal elements, and every branch of countable cofinality has a maximal element.*
- (ii)  *$T$  is countably compact.*
- (iii)  *$T$  is sequentially compact.  $\square$*

A nice application of the coarse wedge topology was found by Gary Gruenhage [11]:

**3.8 Example.** *A locally compact, metalindelöf space which is not weakly  $\theta$ -refinable.*

Let  $S$  be a stationary, co-stationary subset of  $\omega_1$  and let  $T$  be the set of all compact subsets of  $S$ , with the end extension order. Let  $X$  be the chain-completion of  $T$ , with the coarse wedge topology. Then  $X^2 \setminus \Delta$  is as described. This was the first ZFC example of a metalindelöf regular space that is not weakly  $\theta$ -refinable.

The following example has found use in functional analysis. It is attributed to D. H. Fremlin by Richard Haydon [private communication] and has been rediscovered by several researchers, including J. Bourgain, to whom it is attributed by J. Diestel [8, p. 239].

**3.9. Example.** Let  $\omega^*$  stand for the Stone-Ćech remainder of  $\omega$ ; in other words,  $\omega^* = \beta\omega - \omega$ . Let  $\mathcal{C}_0 = \{\omega^*\}$ . Let  $\mathcal{C}_1$  be an uncountable collection of disjoint clopen subsets of  $\omega^*$  whose union is dense. If  $\alpha$  is a successor ordinal and the disjoint collection  $\mathcal{C}_\alpha$  of clopen sets has been defined then  $\mathcal{C}_{\alpha+1}$  is obtained by taking the union of uncountable families of disjoint clopen sets in each member of  $\mathcal{C}_\alpha$ , each family having dense union in its respective member. If  $\alpha$  is a limit ordinal and  $\mathcal{C}_\beta$  has been defined for all  $\beta < \alpha$ , let  $\mathcal{C}_\alpha$  be the collection of all intersections of maximal chains in  $\bigcup\{\mathcal{C}_\beta : \beta < \alpha\}$  and let  $\mathcal{C}_{\alpha+1}$  be the union of (uncountable) families of disjoint clopen sets in each member of  $\mathcal{C}_\alpha$  whose interior is nonempty, each family having dense union in the interior of its respective member. Continue until a limit ordinal  $\gamma$  has been reached such that every member of  $\mathcal{C}_\gamma$  has empty interior, and let  $\mathcal{T}$  be the tree  $\bigcup\{\mathcal{C}_\alpha : \alpha \leq \gamma\}$  ordered by reverse inclusion.

Of course,  $\mathcal{T}$  is chain-complete and rooted, hence compact Hausdorff in the coarse wedge topology. What is especially significant is that it is homeomorphic in a natural way to the decomposition space of  $\omega^*$  whose elements are the closed nowhere dense sets  $F \setminus \bigcup\{C \in \mathcal{C}_{\alpha+1} : C \subset F\}$  as  $F$  ranges over  $\mathcal{C}_\alpha$  and  $\alpha$  ranges over the ordinals  $\leq \gamma$ . [Of course, if  $F \in \gamma$  then  $F$  is nowhere dense and  $\mathcal{C}_{\alpha+1} = \emptyset$ .] The map associating  $F \in \mathcal{T}$  with this nowhere dense subset is a homeomorphism. Moreover, if  $\mathcal{T}$  is a  $\pi$ -base, then the decomposition map from  $\omega^*$  to the decomposition space is irreducible. See [2] for the construction of tree  $\pi$ -bases for  $\omega^*$  and their uses.

The fact that each member of the decomposition space is nowhere dense tells us that no sequence from  $\omega$  will converge anywhere in the compact Hausdorff space which is the quotient space of  $\beta\omega$  formed by identifying the members of the decomposition space to points. So this space shares some of the ‘pathology’ of  $\beta\omega$  and yet the set of nonisolated points is far better behaved, being radial and having lots of convergent sequences. A few other ‘nice’ properties of the remainder will become evident at the beginning of Section 4. These spaces have been studied in an effort to characterize the smallest (uncountable) cardinal  $\kappa$  such that there is a compact Hausdorff space of cardinality  $\kappa$  which is compact but not sequentially compact.

A whole class of even ‘nicer’ compactifications is associated in a natural way to non-Archimedean spaces:

**3.10. Definition.** A collection  $\mathcal{B}$  of subsets of a set is *of rank 1* if, given any two members  $B_1, B_2$ , either  $B_1 \cap B_2 = \emptyset$  or  $B_1 \subset B_2$  or  $B_2 \subset B_1$ . A *non-Archimedean space* is a  $T_0$  [equivalently, Tychonoff] space with a rank 1 base.

A crucial fact about non-Archimedean spaces is that they actually have a base which is a tree under reverse containment ([21]). This makes the proof of such

‘nice’ attributes as ultraparacompactness and suborderability very easy, and also leads in a natural way to embedding them in compact Hausdorff spaces with the coarse wedge topology.

**3.11. Construction.** Given a base  $\mathcal{B}$  for a non-Archimedean space  $X$  such that  $\mathcal{B}$  is a tree under reverse containment, let  $\langle \mathcal{T}, \leq \rangle$  be the chain completion of  $\mathcal{B}$ . For each  $x \in X$  let  $\mathcal{B}(x)$  be the branch of all  $B \in \mathcal{B}$  such that  $x \in B$ . Then the map  $f : X \rightarrow \mathcal{T}$  that takes  $x$  to the supremum of  $\mathcal{B}(x)$  in  $\mathcal{T}$  is easily seen to be an embedding with respect to the coarse wedge topology.

It has long been known that every non-Archimedean space is realizable as some subset of the set of all branches of some tree, endowed with a natural topology analogous to the definition of the Stone space of a Boolean algebra. One simply takes a tree base  $\mathcal{B}$  and proceeds as above; usually, the tree  $\mathcal{T}$  is not explicitly mentioned, only the tree  $\mathcal{B}$  and the set of its branches.

The analogy with the Stone duality goes in the opposite direction, too. Given a tree  $S$ , one can let the space  $\mathcal{X}(S)$  the set of the branches of  $S$ . The resulting space has a tree base  $\mathcal{B}$  in natural association with  $S$ , with  $s \in S$  corresponding to  $B[s] = \{X \in \mathcal{X}(S) : s \in X\}$ . See [21] for details.

Many properties of  $S$  are naturally associated to properties of  $\mathcal{X}(S)$ . For example,  $\mathcal{X}(S)$  is an L-space if, and only if,  $S$  is a Souslin tree [Definition 4.10 below]. One also has some carry-over in Construction 3.11, though one needs to be careful. If  $X$  is an L-space, then every tree-base for  $X$  is indeed a Souslin tree, but its completion  $\mathcal{T}$  will not be an L-space if a finitary tree-base is chosen, since then  $\mathcal{T}$  has uncountably many isolated points. On the other hand, if the base is chosen [as indeed it can be] so that every member, other than an isolated singleton, has infinitely many immediate successors, then  $\mathcal{T}$  will be a compact L-space, as will be shown in a forthcoming paper.

An interesting class of non-Archimedean spaces is provided by trees [and not their branch spaces!] in which each member has at most countably many immediate successors, with the fine wedge topology. For each  $t \in T$  with infinitely many immediate successors, let  $\langle t_n : n \in \omega \rangle$  list them, and let  $B_t^n = V_t \setminus (V_{t_1} \cap \cdots \cap V_{t_n})$ . Each  $t \in T$  with finitely many immediate successors is isolated, so that

$$\{\{t\} : t \text{ is isolated}\} \cup \{B_t^n : n \in \omega, t \text{ has infinitely many immediate successors}\}$$

is a tree base for  $T$  with the fine wedge topology. One consequence of all this is something that foreshadows a theme of the next section:

**3.12. Theorem.** *Every tree in which each element has at most countably many immediate successors is suborderable in the fine wedge topology.*

Indeed, every non-Archimedean space is suborderable. One can also show that every full  $\omega$ -ary tree of limit order height is orderable in the fine wedge topology. C.

Aull [1] took advantage of this to produce a hereditarily paracompact space with a point-countable base with no  $\sigma$ -point-finite base, using the full  $\omega$ -ary tree of height  $\omega_1$ . This idea generalizes to all cardinals in a straightforward way. Incidentally, it is not hard to show that the Michael line is homeomorphic to the full  $\omega$ -ary tree of height  $\omega + 1$  in the fine wedge topology, with the points at level  $\omega$  corresponding to the irrationals. Details will appear in a forthcoming paper.

*From now on, “tree” will always mean, “tree in which every nontrivial pseudo-supremum is a supremum.”*

#### 4. A Short Survey on (Mostly) the Interval Topology

The interval topology has received the lion’s share of attention among set-theoretic topologists as far as topological properties are concerned. Part of the explanation for this is twofold: on the one hand, most of the topologies in Section 2 are not Hausdorff except in trivial cases; and on the other hand, the remaining ones (except for the fantail topology, which coincides with the interval topology on finitary trees) have such strong topological properties that there is far less room for variation than with the interval topology. The following concept highlights this difference:

**4.1. Definition.** A space  $X$  is *monotone normal* (or: *monotonically normal*) if to each pair  $\langle G, x \rangle$  where  $G$  is an open set and  $x \in X$ , it is possible to assign an open set  $G_x$  such that  $x \in G_x \subset G$  so that  $G_x \cap H_y \neq \emptyset$  implies either  $x \in H$  or  $y \in G$ .

[The foregoing is actually a characterization due to C. R. Borges [4] which is very well adapted to our purposes. The usual definition motivates the name “monotone normal” much better.]

Monotone normality is a very strong property. It is hereditary, and it implies both collectionwise normality and countable paracompactness. So the following theorem tells us that trees are ‘very nicely behaved’ in three of the first four topologies:

**4.2. Theorem.** *Every tree is monotonically normal in the coarse wedge, fine wedge, and chevron topologies.*

*Outline of Proof.* For the chevron topology, given  $t \in G$ , let  $G_t = \{t\}$  if  $t$  is isolated, and otherwise let  $G_t = C[s, t]$  for the minimal  $s$  such that  $C[s, t] \subset G$ . For the fine wedge topology,  $G_t$  can be defined by removing from  $V_t$  all of the finitely many  $V_x$  that are not subsets of  $G$  among the immediate successors  $x$  of  $t$ . For the coarse wedge topology, put the choices for the two other topologies together.  $\square$

The well-known Rudin-Balogh characterization of [hereditary] paracompactness in monotonically normal spaces translates very simply to trees in these three topologies: a tree is paracompact iff it does not have a closed copy of an uncountable regular cardinal and hereditarily paracompact iff it has no copies of stationary subsets of uncountable regular cardinals. Since there are no such subspaces in the fine wedge topology at all, we have:

**4.3. Corollary.** *Every tree is hereditarily paracompact in the fine wedge topology.*  
□

The situation is completely different for the interval topology, where monotone normality imposes a very strong structure on the tree: it is equivalent to the tree being a topological direct sum of copies of ordinal spaces (Theorem 4.7 below). This rules out such well-known examples as Aronszajn trees and the Cantor tree.

*For the rest of this article, all topological statements concerning trees will refer to the interval topology.*

Two other characterizations of monotone normal trees are given in the following two definitions.

**4.4. Definition.** Let  $\Lambda$  denote the class of limit ordinals. A tree  $T$  has *Property  $\delta$*  if there exists a function  $f : T \upharpoonright \Lambda \rightarrow T$  such that  $f(t) < t$  for all  $t \in T \upharpoonright \Lambda$ , and such that if  $[f(s), s]$  meets  $[f(t), t]$  then  $s$  and  $t$  are comparable.

**4.5. Definition.** A *neighborset* in a space  $X$  is a function  $U : X \rightarrow \mathcal{P}(x)$  such that  $U(x)$  is a neighborhood of  $x$  for all  $x \in X$ . A neighborset  $V$  *refines*  $U$  if  $V(x) \subset U(x)$  for all  $x \in X$ . A space  $X$  is *halvable* if each neighborset  $U$  of  $X$  has a neighborset  $W$  refining it such that if  $W(x) \cap W(y) \neq \emptyset$  then either  $x \in U(y)$  or  $y \in U(x)$ .

**4.6. Definition.** A subset  $S$  of a tree  $T$  is *convex* if  $[s_1, s_2] \subset S$  whenever  $s_1$  and  $s_2$  are elements of  $S$  satisfying  $s_1 < s_2$ .

**4.7. Theorem.** [22] *Let  $T$  be a tree. The following are equivalent.*

- (1)  $T$  is monotonically normal.
- (2)  $T$  is halvable.
- (3)  $T$  has Property  $\delta$ .
- (4)  $T$  is the topological direct sum of subspaces, each homeomorphic to an ordinal and each convex in  $T$ .
- (5)  $T$  is orderable.
- (6) The neighborhoods of the diagonal in  $T^2$  constitute a uniformity. □

Paracompactness is even more restrictive. Locally compact, paracompact, zero-dimensional spaces are the topological direct sum of compact clopen subspaces. Hence, a tree is easily seen to be paracompact if, and only if, it is a topological direct sum of compact spaces, each homeomorphic to an ordinal. Of course, this implies they are monotone normal. Also, it is easy to see:

**4.8. Theorem.** *The following are equivalent for a tree  $T$ :*

- (1)  $T$  is hereditarily paracompact.
- (2)  $T$  is paracompact and has no uncountable branches.
- (3)  $T$  is the topological direct sum of countable, compact spaces each homeomorphic to an ordinal.
- (4)  $T$  is metrizable. □

And so, most of the topological action here has to do with concepts weaker than monotone normality. Many of these properties have been studied by set-theoretic topologists, but usually only in connection with what are rather cryptically called “ $\omega_1$ -trees”. These are trees of height  $\omega_1$  in which every level is countable. Usually even more conditions are imposed, such as the conditions that every element has successors at all levels of the tree and every element has at least two immediate successors; trees satisfying these latter two properties are often called *normalized*.

Strangely enough, the proofs of most of the general theorems in the literature about topological properties on  $\omega_1$ -trees go through almost verbatim for arbitrary (Hausdorff, by the conventions of these last two sections) trees. One of the rare exceptions is Theorem 4.7 above, where the proof that (3) implies (4) in [16] really does not adapt readily to the general case. In some of the theorems below, however, I will not even add “in effect” when attributing them to various authors, so close is the published proof to one for trees in general. This applies to the following theorem, which introduces an important motif: many familiar topological properties can be reduced to the case where all or all but one of the initial ingredients is an antichain.

**4.9. Theorem.** Fleisner, [10] *Let  $T$  be a tree. The following are equivalent.*

- (1)  $T$  is normal.
- (2) Given a closed set  $F$  and an antichain  $A$  disjoint from  $F$ , there are disjoint open sets  $G$  and  $H$  such that  $A \subset G$  and  $B \subset H$ .  $\square$

Some of the most important classes of trees have definitions involving antichains.

**4.10. Definition.** A tree is *special* if it is a countable union of antichains. A tree is *Souslin* if it is uncountable while every chain and antichain is countable. A tree is *Aronszajn* if it is uncountable while every chain is countable and every level  $T(\alpha)$  is countable.

One of the most useful and obvious topological facts about trees is that every antichain is a closed discrete subspace. A closely related result is:

**4.11. Theorem.** *Let  $X$  be a subset of a tree  $T$ . The following are equivalent:*

- (i)  $X$  is a countable union of antichains.
- (ii)  $X$  is  $\sigma$ -discrete, i.e., it is a countable union of closed discrete subspaces.

*Proof that (ii) implies (i):* It is clearly enough to show that every closed discrete subspace is the countable union of antichains. So let  $D$  be closed discrete, let  $D_0$  be the set of minimal members of  $D$ , and with  $D_n$  defined, let  $D_{n+1}$  be the set of minimal members of  $D \setminus (D_0 \cup \dots \cup D_n)$ . Clearly each  $D_n$  is an antichain of  $T$ . If there were a point  $d$  in  $D$  but not any of the  $D_n$ , then for each  $n \in \omega$  there would be a point  $d_n \in D_n$  such that  $d_n < d$ , and any point in the pseudo-supremum of the  $d_n$  would be in their closure, violating the claim that  $D$  is closed discrete.  $\square$

Thus, in particular, every special tree is a countable union of closed discrete subspaces. This clearly implies every chain is countable, and hence also easily implies that each special tree is developable. This was shown by F. Burton Jones, who gave special Aronszajn trees the name “tin can spaces,” investigating them over a period of many years as candidates for a nonmetrizable normal Moore space, along with the related “Jones road spaces” formed from them in the way described near the end of Section 1. His judgment was partially vindicated when W. Fleissner showed [9] that these spaces are normal under  $MA + \neg CH$ . However, Devlin and Shelah [6] showed that no special Aronszajn tree is normal under  $2^{\aleph_0} < 2^{\aleph_1}$ . Ironically enough, this was the same axiom that Jones used back in 1937 to show the consistency of every separable normal Moore space being metrizable. Thus, in particular, the situation as regards “ $\omega_1$ -Cantor trees” and special Aronszajn trees is exactly parallel: the trees obtained by removing all except exactly  $\omega_1$  points from the top level of the Cantor tree are nonmetrizable Moore spaces, as are special Aronszajn trees;  $MA(\omega_1)$  implies both classes of trees are normal; and  $2^{\aleph_0} < 2^{\aleph_1}$  implies both classes are not normal.

Special trees have another property, which is often given the name “ $\mathbb{Q}$ -embeddability”; but the map involved is almost never a topological embedding, nor is it usually one-to-one. So the following terminology is adopted here:

**4.12. Definition.** Let  $\langle L, \leq_L \rangle$  be a linearly ordered set, and let  $\langle P, \leq_P \rangle$  be a tree. A function  $f : P \rightarrow L$  is called an  $L$ -labeling if it is strictly order preserving: that is,  $p <_P q$  implies  $f(p) <_L f(q)$ . A poset is  $L$ -special if it admits an  $L$ -labeling.

**4.13. Theorem.** *Let  $T$  be a tree. The following are equivalent:*

- (i)  $T$  is special.
- (ii)  $T$  is  $\mathbb{Q}$ -special.
- (iii)  $T$  is  $\sigma$ -discrete in the interval topology.
- (iv)  $T$  is developable in the interval topology.
- (v)  $T$  is subparacompact in the interval topology, and is of height  $\leq \omega_1$ .  $\square$

The proof that (i) is equivalent to (ii) is well known (cf. 9.1 of [28]). The equivalence of (iii) through (v) was demonstrated, in effect, by K.-P. Hart [15] although the class of trees explicitly mentioned was more restrictive.

**4.14. Definition.** Let  $\mathcal{A}$  be a collection of disjoint nonempty sets. A family  $\mathcal{U}$  of sets *expands*  $\mathcal{A}$  if for each  $A \in \mathcal{A}$  there exists  $U_A \in \mathcal{U}$  such that  $A \subset U_A$  and  $B \cap U_A = \emptyset$  if  $B \neq A$ . In case where  $\mathcal{A}$  consists of singletons, we also say  $\mathcal{U}$  *expands*  $\bigcup \mathcal{A}$ .

**4.15. Definition.** A space  $X$  is [*strongly*] *collectionwise Hausdorff* (often abbreviated [*s*]cwH) if every closed discrete subspace expands to a disjoint [*resp.* discrete]

collection of open sets. A space  $X$  is *collectionwise normal* (often abbreviated *cwn*) if every discrete collection of closed sets expands to a disjoint (equivalently, discrete) collection of open sets.

**4.16. Theorem.** (M. Hanazawa [14]) *Let  $S$  be a subspace of a tree. The following are equivalent.*

- (1)  $S$  is *collectionwise Hausdorff* (*cwH*).
- (2) Every antichain of  $S$  expands to a disjoint collection of open sets.
- (3)  $S$  is *hereditarily cwH*.  $\square$

**4.17. Theorem.** (K. P. Hart [16, proof of 2.1], in effect) *Let  $S$  be a subspace of a tree. The following are equivalent:*

- (1)  $S$  is *normal and cwH*.
- (2)  $S$  is *strongly cwH*.
- (3)  $S$  is *hereditarily collectionwise normal*.  $\square$

**4.18. Corollary.** *Every Souslin tree is hereditarily collectionwise normal.*

*Proof.* Every antichain  $A$  is countable and hence is a subset of some clopen initial segment  $T \upharpoonright (\alpha + 1)$ , which is second countable and thus metrizable. Therefore,  $A$  can be expanded to a discrete collection of open subsets of  $T \upharpoonright (\alpha + 1)$ , and hence of  $T$ .  $\square$

**4.19. Corollary.** *The existence of a normal tree that is not collectionwise normal is ZFC-independent.*

*Proof.* If  $\text{MA} + \neg\text{CH}$ , one can either use a special Aronszajn tree (which is not cwH by the Pressing-down Lemma) or an  $\omega_1$ -Cantor tree, as remarked early in Section 4, to give such a tree. On the other hand  $V = L$  implies every locally compact normal space is cwH [29], and so Theorem 4.17 implies it is (hereditarily) cwn.  $\square$

For our next few results, recall that a space is said to be *countably paracompact* [resp. *countably metacompact*] if every countable open cover has a locally finite [resp. point-finite] open refinement.

**4.20. Theorem.** (Nyikos [24]) *Let  $T$  be a tree. The following are equivalent:*

- (1)  $T$  is *countably paracompact* [resp. *countably metacompact*].
- (2) Any countable partition  $\{A_n : n \in \omega\}$  of any antichain of  $T$  expands to a locally finite [resp. point-finite] collection of open sets.  $\square$

**4.21. Corollary.** *Every normal tree is countably paracompact. (“There are no Dowker trees.”)*

*Proof.* Every countable discrete collection of closed sets in a normal space expands to a discrete collection of open sets.  $\square$

**4.22. Corollary.** *Every cwH tree is countably metacompact.*

*Proof.* Given  $\{A_n : n \in \omega\}$  as in 4.20, use cwH to expand the antichain that is their union to a disjoint collection of open sets, and let  $U_n$  be the union of the ones that meet  $A_n$ .  $\square$

**4.23. Theorem.** *The existence of a countably paracompact tree that is not cwH is ZFC-independent.*

*Proof.* W. S. Watson showed that under  $V = L$ , every locally compact, countably paracompact space is cwH [30]. On the other hand, a  $\omega_1$ -Cantor tree is not cwH, but is normal under  $\text{MA} + \neg\text{CH}$ , and hence countably paracompact under the same axiom because it is normal and Moore.  $\square$

The following three questions are related to the last three results. Note the contrast in the phrasing as to set-theoretic status.

**4.24. Problems.** *Is there a ZFC example of a cwH tree that is (a) not countably paracompact or (b) not normal or at least (c) not monotone normal?*

*Caution.* K. P. Hart credits S. Todorćević with having even shown, assuming “an at least inaccessible cardinal”, that it is consistent for every cwH tree to be orderable [16]. If this had been correct as stated, these problems would be solved modulo inaccessibles, but “tree” referred to  $\omega_1$ -trees only.

If there is a Souslin tree, there is a cwH tree which is not countably paracompact and hence (by 4.21) not normal. Details will appear in [24]. Earlier, Devlin and Shelah [7] used the stronger axiom  $\diamond^+$ , a consequence of  $V = L$ , to construct a cwH non-normal tree which is  $\mathbb{R}$ -special, hence not countably paracompact (see Corollary 4.40 below).

**4.25. Problem.** *Is it true in ZFC that every countably paracompact cwH tree is (collectionwise) normal?*

**4.26. Problems.** *Is it ZFC-equiconsistent that every countably paracompact tree is (a) normal? (b) collectionwise normal?*

The last question is phrased the way it is because of a gap in our consistency results. On the one hand, a  $\Delta$ -set of real numbers that is not a  $Q$ -set gives a countably paracompact non-normal tree, and such sets of reals are consistent assuming just the consistency of ZFC [17]. On the other hand, the only known models in which every locally compact, countably paracompact space is *strongly* cwH require large cardinal axioms. This also applies to “first countable” in place of “locally compact,” and we know of no axioms which give normality in countably paracompact trees without also giving cwH. In fact, the following problem is of interest quite apart from its obvious applicability to Problem 4.26:

**4.27. Problem.** *Does  $V = L$  or some other ZFC-equiconsistent axiom imply that every locally compact, or every first countable, countably paracompact regular space is strongly  $cwH$ ?*

Unlike countable paracompactness, countable metacompactness has generally been thought of as a very weak property. However, the following suggests that its failure for trees is a fairly ordinary occurrence:

**4.28. Example.** (Nyikos [24]) *The full binary tree of height  $\omega_1$  is not countably metacompact.*

There are even trees that are  $\mathbb{R}$ -special, yet not countably metacompact, such as the tree of all ascending sequences of rational numbers [24], designated  $\sigma\mathbb{Q}$  in [28].

Being  $\mathbb{R}$ -special turns out to be a fairly strong “generalized metric” property for trees. It is easily shown to imply quasi-metrizability, but is much stronger [23], and we also have:

**4.29. Theorem.** (K. P. Hart, [15]) *Let  $T$  be a tree. The following are equivalent.*

- (1)  *$T$  is  $\mathbb{R}$ -special.*
- (2)  *$T$  has a  $G_\delta$ -diagonal.*
- (3) *The set of nonisolated points of  $T$  is a  $G_\delta$ .*
- (4) *The set of isolated points of  $T$  is a countable union of antichains.  $\square$*

In the same article, Hart also showed the remarkable fact that every finitary  $\mathbb{R}$ -special tree is special. So, for example, if each element of  $T$  has  $\leq \mathfrak{c}$ -many immediate successors, and we add a full binary tree of height  $\omega$  between each point and its immediate successors, then  $T$  embeds as a closed subspace in the resulting tree, and if  $T$  is special, so is the resulting tree. On the other hand, if  $T$  is  $\mathbb{R}$ -special but not special, the resulting tree will be quasi-metrizable but not special [23]. Much is still unknown about quasi-metrizable trees, including:

**4.30 Problem.** *Is it consistent that every tree without uncountable branches is quasi-metrizable?*

This problem is worded the way it is because  $\omega_1$  embeds in every tree with an uncountable branch, and is not quasi-metrizable; and because a Souslin tree is not quasi-metrizable [23].

Condition (4) in Theorem 4.29 was an ingredient in the proof of:

**4.31. Theorem.** (Nyikos [24]) *Let  $T$  be a tree. The following are equivalent:*

- (1)  *$T$  is perfect; that is, every closed subset of  $T$  is a  $G_\delta$ .*
- (2)  *$T$  is  $\mathbb{R}$ -special, and every antichain is a  $G_\delta$ .  $\square$*

The fact that (1) implies (2) was essentially shown by M. Hanazawa [14] who used it to help answer a question of K. P. Hart [15]: “Is every  $\omega_1$ -tree with a  $G_\delta$ -diagonal perfect?” The answer is affirmative under  $MA + \neg CH$  as Hart himself pointed out, but it is negative under the axiom  $\diamond^*$  with the help of which Hanazawa constructed a counterexample. [*Caution.* The example, an Aronszajn tree, is claimed in [13] to be countably metacompact, but it is not.] The paper also showed the following for  $\omega_1$ -trees:

**4.32. Corollary.** *Every collectionwise Hausdorff,  $\mathbb{R}$ -special tree is perfect.*

*Proof.* Because  $T$  is  $\mathbb{R}$ -special, its height is  $\leq \omega_1$ . Hence, by cwH, every antichain expands to a disjoint family of countable open sets, and hence is a  $G_\delta$ .  $\square$

Under  $MA + \neg CH$ , we can weaken the hypothesis and strengthen the conclusion:

**4.33. Theorem.** (Nyikos [22]) *If  $MA + \neg CH$ , a tree is metrizable if, and only if, it is collectionwise Hausdorff and has no uncountable chains.*

A related ZFC result is:

**4.34. Theorem.** (Nyikos [22]) *A tree is metrizable if, and only if, it is special and cwH.*

Another corollary of Theorem 4.31 is:

**4.35. Corollary.** *If  $V = L$ , or PMEA, then the following are equivalent for a tree  $T$ :*

- (1)  $T$  is perfect.
- (2)  $T$  is  $\mathbb{R}$ -special and countably metacompact.

*Proof.* As is well known, every perfect space is countably metacompact, so from 4.31 it follows that (1) implies (2). Conversely, if  $V = L$ , then every closed discrete subspace in a locally countable, countably metacompact space is a  $G_\delta$  [20]; and every  $\mathbb{R}$ -special tree is of height  $\leq \omega_1$  and hence locally countable. Under PMEA, every closed discrete subspace in a first countable, countably metacompact space is a  $G_\delta$  (D. Burke, [5]) and (2) similarly implies (1).  $\square$

I do not know whether the set-theoretic hypotheses in 4.35 can be dropped. More generally:

**4.36 Problem.** *Is every closed discrete subset of a countably metacompact tree of height  $\leq \omega_1$  a  $G_\delta$ ?*

The following is a pleasing counterpoint to Theorem 4.31:

**4.37. Theorem.** (Nyikos [24]) *Let  $T$  be a tree. The following are equivalent.*

- (1)  *$T$  is perfectly normal.*
- (2) *Every closed subset of  $T$  is a regular  $G_\delta$ ; that is, it is of the form  $\bigcap \{cl(U_n) : n \in \omega\}$  where each  $U_n$  is open.*
- (3)  *$T$  is  $\mathbb{R}$ -special and every antichain is a regular  $G_\delta$ .  $\square$*

Perfect normality is a highly axiom-sensitive property where trees are concerned. Under  $V = L$ , normal trees are collectionwise normal, and M. Hanazawa [14] used the consequence  $\diamond^*$  of  $V = L$  to construct a perfectly normal Aronszajn tree which is, of course, not special. Under  $MA + \neg CH$ , the cwH ones are all metrizable (Theorem 4.32), but every Aronszajn tree is a nonmetrizable example, and special, as is every  $\omega_1$ -Cantor tree. Under  $2^{\aleph_0} < 2^{\aleph_1}$  and EATS (“Every Aronszajn tree is special”) no Aronszajn tree can be normal (cf. [6] or [27]), nor can any  $\omega_1$ -Cantor tree, by the Jones Lemma. However, there is an axiom compatible with CH under which there is a perfectly normal non-cwH tree of height  $\omega + 1$  [22]. Finally, if strongly compactly many random reals are added to a model of  $MA + \neg CH$ , Theorem 4.32 still holds in the forcing extension, but PME<sub>A</sub> also holds and so every normal, first countable tree is cwn, hence every perfectly normal tree is metrizable. However, the following may still be open:

**4.38. Problem.** *Is it ZFC-equiconsistent for every perfectly normal tree to be metrizable?*

Weakening normality slightly to countable paracompactness, we have:

**4.39. Theorem.** (Nyikos [24]) *Every  $\mathbb{R}$ -special, cwH, countably paracompact tree is (collectionwise) normal.  $\square$*

**4.40. Corollary.** *If  $V = L$ , every  $\mathbb{R}$ -special, countably paracompact tree is collectionwise normal.*

*Proof.* By 4.39 and the proof of 4.24.  $\square$

Thus Problems 4.25 and 4.26 have affirmative answers for  $\mathbb{R}$ -special trees.

**4.41. Problems.** *Is there a ZFC example of a tree which is not special but is (a) perfect or (b) countably metacompact and has no uncountable branches?*

The search for ZFC examples is made difficult by the fact that  $\sigma\mathbb{Q}$ , which is the simplest ZFC example of an  $\mathbb{R}$ -special, non-special tree of which I am aware, is not countably metacompact. Consistent examples have long been known, like the Devlin-Shelah  $\diamond^*$  example mentioned earlier: a cwH (hence non-special, by the Pressing-Down Lemma) Aronszajn tree which is not normal but is  $\mathbb{R}$ -special, hence perfect. In the same paper [7], they defined:

**4.42. Definition.** An Aronszajn tree is *almost Souslin* if every antichain meets a nonstationary set of levels.

They showed that an Aronszajn tree is cwH iff it is almost Souslin; this is an easy application of Theorem 4.11 and the Pressing-down Lemma.

Back in the 1980's, Hanazawa did some extensive cataloguing of how Aronszajn trees behave under  $V = L$ . He constructed or listed examples exhibiting all combinations of the properties considered in this section and not ruled out by the results mentioned or proven here — except one: we still do not know whether there is a countably paracompact, non-normal Aronszajn tree under  $V = L$ . He also has catalogued their behavior with respect to some properties not mentioned here, cf. [12].

Finally, we return to the property with which we began this section.

**4.43. Problems.** *Is it consistent that every (a) normal or (b) every collectionwise normal tree is monotonically normal?*

Recall also Problem 4.24 (c), which can be stated negatively: is it consistent that every cwH tree is monotonically normal? Here is a partial result:

**4.44 Theorem.** (Nyikos [22]) *If  $MA + \neg wKH$ , then every cwH tree of height  $< \omega_2$  is monotonically normal.*

[Compare Theorem 4.33.] Here “wKH” refers to the existence of weak Kurepa trees. Once informally called “Canadian trees,” these are trees of height and cardinality  $\omega_1$  that have more than  $\aleph_1$  uncountable branches. Of course, the full binary tree of height  $\omega_1$  is a weak Kurepa tree under the continuum hypothesis. So the axiom in 4.44 negates CH, and it is known to imply that there are inaccessible cardinals.

It would be interesting to see whether large cardinals are really needed in 4.44. Its proof goes through just on the assumption of “every weak Kurepa tree has a special Aronszajn subtree,” whose consistency is apparently not known to depend on large cardinal axioms.

One of the ingredients in the proof of 4.44 is of independent interest. Call a tree  $\sigma$ -orderable if it is the countable union of closed, orderable subtrees.

**Theorem 4.45.** (Nyikos [22]) *A tree is orderable if, and only if, it is  $\sigma$ -orderable and cwH.  $\square$*

Incidentally,  $\sigma$ -orderability is easily seen to be equivalent to a property to which J. E. Baumgartner tried to transfer the term “special” [3]. His usage does not, however, seem to have caught on, and “ $\sigma$ -orderable” seems to be a good a name as any for Baumgartner’s property.

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