SEQUENTIAL EXTENSIONS OF COUNTABLY COMPACT SPACES

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ABSTRACT. The first known examples of subsequential countably compact Hausdorff (T_2) spaces that are not sequential are given here, including one that is Tychonoff under CH. The sequential extensions of such spaces cannot be T_2, but the extensions we construct are T_1. The problem of whether it is consistent for there to be a compact T_2 subsequential, non-sequential space is discussed. It is shown that an affirmative answer would also solve the old problem of whether it is consistent for there to be a compact non-sequential T_2 space in which every countably compact subset is closed.

We also give the first known example of an infinite subsequential, countably compact T_1 space with no nontrivial convergent sequences. The main tool in all the constructions is a base matrix tree of subsets of ω; in other words, a collection of subsets of ω whose Stone-Čech remainders form a tree π-base in $\beta\omega \setminus \omega$.

1. Introduction

A major theme in many branches of mathematics is that of extensions of structures. Think of Galois field theory, analytic continuation in complex analysis, and the concept of Ext in module theory, to name but a few examples. In general topology the most extensively researched example is that of Hausdorff compactifications of Tychonoff spaces. Another example is that of connectification: the study of how "nice" a connected space containing a given space can be.

Another example is the study of sequential and pseudo-radial (a.k.a. chain-net) extensions of spaces. The latter spaces are those in which the the closure of a set A is found by iterating the process of adjoining limits of well-ordered nets. In the sequential case there is an obvious restriction: the space must be **countably tight**; that is, if a point p is in the closure of a subset A then there must be a countable subset B of A such that $p \in \overline{B}$. [As usual, overhead bars stand for

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closure.] This is because **sequential** spaces are characterized by the fact that the closure of a set is found by iterating the process of adjoining limits of convergent sequences, and because every subspace of a countably tight space is countably tight. In contrast, Martin Sleziak [S] has shown that every topological [*resp.* T_0, T_1] space can be embedded in a pseudo-radial [*resp.* T_0, T_1] space. Earlier, Jinyuan Zhou [Zh] had given a different construction under $\mathfrak{p} = \mathfrak{c}$, embedding any T_1 space of countable tightness in a pseudo-radial space. But not every countably tight space can be embedded in a sequential space: easy ZFC examples can be found in [FR]. The following related problem from [FR] is still unsolved.

Problem 1. Is every subsequential compact T_2 space sequential?

A space is here called **subsequential** if it can be embedded in a sequential topological space. In [FR] an easy example is given of a compact T_1 subsequential space that is not sequential, but it is also noted that any counterexample to Problem 1 would require extra set-theoretic axioms, since the PFA implies that every compact T_2 countably tight space is sequential. Franklin and Rajagopalan go on to give examples of pseudocompact Tychonoff spaces which can be embedded in sequential T_2 spaces, and explain why every countably compact subspace of a sequential T_2 space is itself sequential. Implicit in this is the question of whether every subsequential countably compact T_2 space is sequential. This question was made explicit in [Ny]. One of the aims of this article is to show that the answer is negative [Examples 2.2 and 2.3.], but as we have just said, the extension space cannot itself be T_2.

Since most general topologists confine themselves to Tychonoff spaces, a few comments may be in order as to why we bother to construct such examples. In the first place, we are only dealing with extensions of countably compact spaces, and countably compact (and especially compact) T_1 spaces are much better behaved than T_1 spaces in general. Recall, for example, Gryzlov's extension of Arhangelskii's celebrated theorem to T_1 spaces: every compact first countable T_1 space is of cardinality $\leq \mathfrak{c}$. Less well known but still striking is Norman Levine's theorem that every compact space [no separation axioms assumed!] of cardinality $\aleph_1$ is sequentially compact. This is an obvious consequence of the second of the following recent theorems, which again assume no separation axioms.

Theorem A. [AW] *Every countably compact space of hereditarily Lindelöf degree* $< \mathfrak{t}$ *is sequentially compact.*

Theorem B. [Ny2] *If X is a compact space with a network $\mathfrak{N}$ of cardinality $\leq \mathfrak{t}$, such that every point of X is in fewer than $\mathfrak{t}$ members of $\mathfrak{N}$, then X is sequentially compact.*

Theorem C. [Ny2] *Every countably compact space of cardinality* $< \mathfrak{h}$ *is sequentially compact.*

Theorem D. [Ny2] *If every splitting tree has a chain of length $\mathfrak{h}$, then every compact space of cardinality $\leq \mathfrak{h}$ is sequentially compact.*

The cardinal $\mathfrak{t}$ is, as in [vD] and [V], the least cardinality of a complete tower on ω. Closely related is $\mathfrak{h}$, the least height of a splitting tree on ω. [These concepts are defined in Section 2.] A good reference to $\mathfrak{h}$ is [BPS].

The above theorems are relevant to Examples 2.4, 2.9, and 3.1, which are sequential T_1 extensions of countably compact, Hausdorff spaces that are not sequentially compact (hence not sequential). Examples 2.4 and 2.9 are constructed in a unified fashion along with a T_1 pseudo-radial extension of a countably compact 0-dimensional space (Example 2.1) which has a nicer structure than the more general constructions mentioned above.

Example 3.1 is quite different. It is a countably compact sequential scattered (hence T_1) space X with a countably compact subspace Y which has no nontrivial convergent sequences at all. No subsequetial Y with this stronger property can be Hausdorff. The whole space X is locally countable, weakly first countable, and of scattered height and sequential order ω_1. In fact, the Cantor-Bendixson level of each point is the same as its sequential order.

Theorems A through D warn us not to expect the examples to be "very small." Even more to the point is the following theorem, a rephrasing of Theorem 1 of [A].

Theorem E. *Let Y be a subsequential T_2 space and let y be a nonisolated point of Y. If X is a countably compact T_2 space containing Y, then there is a nontrivial sequence in X converging to y.*

To prove this theorem, Elena Aniskovič used a penetrating analysis of the convergence structure of a subsequential space. The sequential extension of Y, even though it need not satisfy any of the usual separation axioms, still exerts a strong influence on which filters on Y converge in Y and even in X. Franklin and Rajagopalan [FR] showed this in a somewhat different way through the use of quotient maps.

An immediate corollary of Theorem E is that every countably compact subsequential T_2 space contains a nontrivial convergent sequence. This shows that none of the usual (consistent) compact T_2 countably tight non-sequential spaces are subsequential. Theorem E is also relevant to Example 3.1, in which a countably compact subsequential T_1 space is constructed in which every convergent sequence is trivial; such an example could not be T_2 by what we have just seen. This also follows from another corollary of Theorem E:

Corollary. *In a subsequential T_2 space, every countably compact subset is closed.*

This corollary is proven in Section 4, where it is also explained how it sheds light on just how difficult Problem 1 is, unless there is somehow an easy positive answer in ZFC.

Section 2. A unified trio of constructions

Each of the three examples in this section features a pseudo-radial (a.k.a. chain-net) T_1 space X with a countably compact Hausdorff subspace Y that is not sequentially compact. In Example 2.1, X and Y are compact and Y is 0-dimensional. Example 2.4 is a subspace of a special case of Example 2.1, and exists if CH is assumed; in it, X is countably compact and sequential. In the natural ZFC generalization of Example 2.4, X is just pseudo-radial. To make X sequential in ZFC (Example 2.9), I had to sacrifice regularity of Y while using the same underlying set with a finer topology.

These three examples have ω as a dense subspace, and disjoint subspaces D and T, each with underlying set the same (up to order isomorphism) $\mathfrak{c}$-ary tree. In Example 2.4 this is the full $\mathfrak{c}$-ary tree of height ω_1. In all three examples, the relative topology on D is the interval topology. In Examples 2.1 and 2.4, the relative topology on T is the coarse wedge topology, which has as a base the Boolean algebra generated by all wedges $t^{\uparrow} = \{s \in T : t \leq s\}$ such that t is not on a limit level of T. In Example 2.9 we add the wedges $t^{\uparrow}$ where t is a limit ordinal of uncountable cofinality to the set of generators. In all these examples, the topology on $D \cup T$ is an example of what Steve Watson calls a resolution of T, a generalization of the Alexandroff duplicate.

The subspace $Y = \omega \cup T$ in Example 2.1 is a special case of a construction that uses a *splitting tree* of subsets of ω. A set S is said to **split** a set A if both $A \cap S$ and $A \setminus S$ are infinite. A **splitting family on** ω is a family of subsets of ω such that every infinite subset of ω is split by some member of the family. The family is called *a splitting tree* if it is a tree by reverse almost inclusion. The least cardinality of a splitting family is denoted $\mathfrak{s}$, while least height of a splitting tree is denoted $\mathfrak{h}$. It is easy to show that $\omega_1 \leq \mathfrak{t} \leq \mathfrak{h} \leq \mathfrak{s}$. Dordal [D] constructed models of ZFC in which $\mathfrak{t} = \omega_1$ and $\mathfrak{h}$ is an arbitrarily high aleph.

The construction of Y in Example 2.1 was independently discovered by Fremlin, Bourgain, myself, probably Petr Simon, and perhaps others. There is some question about who was the first to do it, Fremlin or Bourgain, with Haydon favoring the former and Diestel the latter. Our construction works for any splitting tree, but the proof is simpler if we use a *base matrix tree*, a particular kind of splitting tree such that every infinite subset of ω almost contains some some member of the tree. The existence of base matrix trees in ZFC is a deep result of Balcar, Pelant and Simon [BPS].

In Example 2.1, we let T be the order completion of a tree that indexes the members of a base matrix tree of minimal height. The order completion of a tree is defined by giving every downwards closed chain a unique supremum if it does not already have one. In the case of a splitting tree, every chain without a greatest element fails to have a unique supremum; if it is bounded above, the part of the tree above it has a collection of uncountably many minimal members, which can and will be taken to be $\mathfrak{c}$-many. Thus, if we let $T(\alpha)$ represent the αth level of T and let $\Lambda(T) = \bigcup\{T(\alpha) : \alpha \text{ is a limit ordinal}\}$, then the splitting tree consists of $\{A_t : t \in T \setminus \Lambda(T)\}$. In examples 2.4 and 2.9, all maximal members are removed from the order completion to produce T.

2.1. Example. Let T be the order-completion of a base matrix tree in which each non-minimal member has $\mathfrak{c}$-many immediate successors. To each t in T, associate a new point $d_t = \langle t, 1\rangle$ and let $D = \{d_t : t \in T\}$. If t is not on a limit level, basic neighborhoods of d_t are the sets of the form $\{d_t\} \cup A_t \setminus F$ where F is a finite subset of ω. Thus $\{d_t\} \cup A_t$ is the one-point compactification of A_t. Of course, if $s > t$ then $A_s \subset^* A_t$ and so the space $\omega \cup D$ fails to be Hausdorff in a big way.

If $t \in \Lambda(T)$, then basic neighborhoods of d_t are the sets of the form $D(s, t] = \{d_x : s < x \leq t\} \cup A_s \setminus F$. As before, F is a finite subset of ω. It is easy to see that this makes the relative topology on D the interval topology it acquires as a tree.

Now we are ready to define the neighborhoods of $t \in T$ in the whole space $X = \omega \cup D \cup T$. For each $t \in T$ let $V_\emptyset(t) = t^\uparrow \cup \{d_s : s \in t^\uparrow\} \cup A_t$. For each finite set of (not necessarily immediate) successors $s_1, \dots, s_n$ of t, let $S = t^\uparrow \setminus (s_1^\uparrow \cup \dots \cup s_n^\uparrow)$ and let

$$V_t(s_1, \dots, s_n) = S \cup \{d_s : s \in S\} \cup [A_t \setminus (A_{s_1} \cup \dots \cup A_{s_n})].$$

If t is not on a limit level of T, its basic neighborhoods are of the form $V_t(s_1, \dots, s_n) \setminus F$ where F is a finite set that does not include t, and the s_i are immediate successors of t.

If t is on a limit level then its basic neighborhooods are of the form

$$V_x(s_1, \dots, s_n) \setminus (F \cup \{d_s : s \in t^\downarrow\})$$

where x is on a successor level and $x < t$, and the s_i are immediate successors of t (not of x). It is important to omit the members of D indexed by $t^\downarrow$ because these are in the closure of any infinite subset of ω that is indexed by some s on a successor level above t. Failure to omit them would mean that s and t do not have disjoint open neighborhoods in $Y = \omega \cup T$.

It follows from this description that the relative topology on T is the coarse wedge topology. It is also easy to see that $Y = \omega \cup T$ is Hausdorff (indeed, 0-dimensional). Because the A_s indexed by the immediate successors of t are a MAD family of subsets mod finite of A_t, no sequence from ω converges to any point of T. On the other hand, every infinite subset of ω has uncountably many points of T in its closure: each $S \in [\omega]^\omega$ almost contains some A_t and with it every A_s, $s > t$, so that the whole of $V_t(\emptyset)$ is in the closure of S except perhaps for a finite subset of ω.

Compactness of T is part of the basic theory of the coarse wedge topology [Ny1, Theorem 3.4], and compactness of $\omega \cup T$ is an easy consequence given the above description of the basic nbhds of points of T. The rest follows quickly from two lemmas:

2.2. Lemma. *$D \cup T$ is radial; that is, if $x \in \overline{A}$ then there is a well-ordered net in A converging to x.*

2.3. Lemma. *If $t \in T, S \subset \omega$ and $t \in \overline{S}$, then $t \in \overline{(\overline{S} \cap D)}$.*

From these two lemmas it follows that X is pseudo-radial, of order 2. In fact, if $S \subset \omega$ and $t \in \overline{S}$ then there is a well-ordered net from $\overline{S} \cap D$ converging to t, while $d_t \in \overline{S} \iff S \cap A_t$ is infinite $\iff$ any sequence that lists $S \cap A_t$ converges to d_t.

Proof of Lemma 2.2. For D this is trivial: the neighborhood $\{d_s : s \in t^{\downarrow}\}$ is a copy of an ordinal. In [Ny1] it is shown that every tree is radial in the split wedge topology, which coincides with the coarse wedge topology for trees that are order-complete. A minor adaptation of this proof shows that every point $t \in T$ in the closure of a subset S of D is the limit of a well-ordered net from D. Specifically, if t is not on a limit level, then t is in the closure of S iff S meets infinitely many basic $V_s(\emptyset)$ based on immediate successors of t; and then every choice function with domain ω for infinitely many of these $V_s(\emptyset)$ converges to t. If t is on a limit level, then either the same thing occurs, or else there is a well-ordered net $\langle s_\xi : \xi < \alpha \rangle$ in T converging up to t from below, such that $S \setminus (t^{\uparrow} \cup t^{\downarrow})$ meets V_{x_ξ} for all ξ. But every neighborhood of t contains $V_{x_\xi} \setminus t^{\uparrow}$ for cofinally many $\xi < \alpha$. So another choice function gives a well-ordered net from S converging to t. $\square$

Proof of Lemma 2.3. The preceding proof can be modified to characterize those $S \subset \omega$ that have t in their closure. Simply replace "meets" with "hits," i.e., "meets in an infinite set." Then we get a family of appropriately situated d_x having t in their closure, with each d_x in the closure of S. The only part that needs special attention is the last case, where S hits only sets of the form $V_{x_\xi} \setminus t^{\uparrow}$. But in this case, if $A_x \subset S$ then $x \notin t^{\downarrow}$. The sets A_x come from a base matrix tree and so there are enough well situated d_x in this case too. $\square$

Examples 2.4 and 2.9 use a subset of Example 2.1 formed by removing the top-most points ("leaves") of T and D. In the relative topology, this gives us a countably compact pseudo-radial space. If T was of height $\omega_1 + 1$ the resulting subspace of X is sequential; this is our second example.

2.1. Example. Let T be the full $\mathfrak{c}$-ary tree of height $\omega_1 + 1$, and let $\Lambda(T)$ denote the points of T on limit levels. We invoke CH to index a index a base matrix tree by $T \setminus \Lambda(T)$. More generally, we could use the axiom $\mathfrak{h} = \omega_1$, because the least height of a base matrix tree is also the least height of a splitting tree [BPS]. The individual levels of $T \setminus \Lambda(T)$ then index MAD families of subsets of ω.

In the particular case of Example 2.1 that results from T, we let X be the subspace $\omega \cup C \cup S$ where S is the full binary tree of height ω_1 and C is the corresponding subset of D, and let Y be the subspace $\omega \cup S$. Removal of the topmost points of D and T does not affect the argument that every point of ω has a cluster point in Y, nor the argument that no sequence in ω can converge to a point in Y.

2.5. Theorem. *X is sequential of order 2 and countably compact, and Y is countably compact.*

This theorem is an easy consequence of the foregoing remarks and of the following lemma.

2.6. Lemma. *$C \cup S$ is countably compact and Fréchet-Urysohn.*

This lemma in turn follows easily from the next two:

2.7. Lemma. *A rooted tree is Hausdorff in the coarse wedge topology iff it is a semilattice (equivalently, a complete lattice) with respect to greatest lower bound.*

2.8. Lemma. *A Hausdorff tree is countably compact in the coarse wedge topology iff it has finitely many minimal elements, and every branch (i.e., maximal chain) of countable cofinality has a greatest element.*

Since S satisfies all the hypotheses in these two lemmas, it is countably compact. Lemma 2.7 is trivial, while the second conclusion in Lemma 2.6 is clear from Lemma 2.2 and the fact that every point of S has only countably many predecessors.

Proof of Lemma 2.8. Necessity is clear. Conversely, let A be an infinite subset of S. If A has an infinite chain, then its supremum is a limit point of A. If not, s be the g.l.b. of A. There are two elements a_0, b_0 in A whose g.l.b. is s [Ny1, Theorem 3.2]. If there is an an infinite subset B of A such that all pairs in B have s as their g.l.b., then any 1-1 sequence in B converges to s. If not, we can inductively define elements $s_{n+1} > s_n$ beginning with $s_0 = s$, and infinite subsets $A_{n+1} \subset A_n$ with $A_0 = A$ and $a_n \in A_n$ such that g.l.b.$(a_n, a_m) = s_n$ whenever $n < m$. Then $a_n \to sup_n s_n$. This supremum exists since the s_n are bounded above, and S is Hausdorff. □

Lemma 2.6 now follows by applying the same argument to subsets of D, to get every infinite subset A of D a limit point in T *unless* A has an infinite chain. In this case, A has a limit point in D itself.

2.9. Example. If $\mathfrak{h} > \omega_1$ then something needs to be done about the points of D and T on limit levels of uncountable cofinality. The ones in D can be omitted without affecting the countable compactness argument, as can the leaves of T.

However, the others cannot be removed without destroying countable compactness: each t like this has uncountably many immediate successors, and these would no longer have a limit point. What we do instead is to refine the topology by adding sets defined like $V_x(s_1, \dots s_n) \setminus \omega$ to the topology as a *weak base at* t. That is, if $cf(ht(t)) > \omega$ we let $\mathcal{Z}(t)$ be the collection of sets of the form:

$$Z_t(s_1, \dots s_n; F) = [t^\uparrow \setminus (s_1^\uparrow \cup \dots \cup s_n^\uparrow)](= S) \cup \{d_t : s \in S\} \setminus F \ (F \text{ finite, } t \notin F)$$

and declare a set U to be open iff it contains a member of $\mathcal{Z}(t)$ for each t in U at a limit level of uncountable cofinality and is a neighborhood (in the original topology) of every other point it contains.

Because X is T_1, this is equivalent to t being in the closure of a $H \subset X \setminus \{t\}$ iff it is in the closure of the members of $Z_t(s_1, \dots s_n; F) \setminus \{t\}$ that are themselves in the closure of H. In $Z_t(s_1, \dots s_n; F)$, all points except t have neighborhoods in the original topology that meet $X \setminus \omega$ in a subset of $Z_t(s_1, \dots s_n; F)$ itself. Thus we need only add points of ω to expand $Z_t(s_1, \dots s_n; F)$ to make an open neighborhood of t. Now Lemma 2.3 continues to hold in this finer topology (in fact, there are fewer cases to consider) and so it follows as with Example 2.4 that X is sequential. The proof that Y is countably compact in this topology is substantially the same as with Example 2.4.

In the (very common!) models where $\mathfrak{t} = \mathfrak{h}$, the sets of the form $Z_t(s_1, \dots s_n; F) \cup A_x$ form a base for the neighborhoods of t. This is because, if a subset I of ω meets A_x for all $x < t$, then I will also hit infinitely many sets of the form A_s where s is an immediate successor of t: were it not so, we could subtract off finitely many A_s from I, and then the sets A_x would trace a complete tower of cofinality $\geq \mathfrak{t}$ on what is left of I, contradicting $\mathfrak{t} = \mathfrak{h} = ht(T)$. And now it follows that every neighborhood of t meets I.

Similarly, if $I \subset \omega$ and $t \in \overline{I}$, then t has infinitely many immediate successors s such that I has a subsequence converging to $d_{s'}$ for some $s' \geq s$. If $\mathfrak{t} < \mathfrak{h}$ then the neighborhoods of t are more complicated, but if t does not have infinitely many successors as described just now, then some set of the form $Z_t(s_1, \dots s_n; F) \cup (\omega \setminus I)$ is an open neighborhood of t missing I.

Section 3. An example with no nontrivial convergent sequences.

3.1. Example. This example is built by transfinite induction on the countable ordinals, one level at a time, levels alternating in their basic description. Begin with $\omega = Y_{-1}$ and let T index a base matrix tree on ω. Let $D_0 = \{d_t : t \in T\}$ and let a base of neighborhoods for each d_t be defined as before: it consists of all sets of the form $\{d_t\} \cup A_t \setminus F$ where F is a finite subset of A_t. Then Y_0 is also indexed by T, and for each y_t we pick a set S_t of denumerably many δ_s indexed on the level of T immediately above t, and let a weak base at y_t consist of all sets of the form $\{y_t\} \cup S_t \setminus F$ where F is a finite subset of S_t.

After this, if Y_β has been defined, and $\alpha = \beta + 1$, we let $\mathcal{M}_\alpha$ be a MAD family of countable subsets of Y_α, while if α is a limit ordinal and D_ξ and Y_ξ have been defined for all $\xi < \alpha$, we let $\mathcal{M}_\alpha$ be a MAD family of countable closed discrete subspaces of $\bigcup\{Y_\xi : \xi < \alpha\}$. In either case, for each $M \in \mathcal{M}_\alpha$, let T_M index a base matrix tree on M, and let $T_\mathcal{M}$ be the union of all the T_M ($M \in \mathcal{M}_\alpha$) with the direct sum order. Let $D_\alpha = \{d_t : t \in T_\mathcal{M}\}$ and let a weak base for each d_t be defined as the base was defined for $d_t \in M_0$. Y_α and the weak bases of its points are defined just as they were for $\alpha = 0$, with T_M replacing T.

Let $Y = \bigcup\{Y_\alpha : -1 \leq \alpha < \omega_1\}$ and let $Z = Y \cup \bigcup_{\alpha<\omega_1} D_\alpha$. Since Z is weakly first countable, it is sequential. It is T_1 and scattered; in fact, it has ω as a dense set of isolated points, and it is easy to see that $Z_\alpha = \bigcup\{D_\xi \cup Y_\xi : \xi < \alpha\}$ is open in Z for all $\alpha \in \omega_1$ and that each Y_α and D_α is discrete in its relative topology, with Y_α closed in $Z_{\alpha+1}$ and D_a closed in $Z_\alpha \cup D_\alpha$. As defined here, Z is not countably compact, but it is easy to extend it to a weakly first countable (hence sequential), countably compact T_1 space.

Claim. Y is a countably compact subspace in which every convergent sequence is eventually constant.

Proof of claim. Countable compactness is proven similarly to the previous examples, as follows. Let $\{y_n : n \in \omega\}$ be an infinite subset of Y. We may assume that either there exists α such that $y_n \in Y_\alpha$ for all n or else that y_{n+1} is in a later Y_α than is y_n. In the latter case, $\{y_n : n \in \omega\}$ is a closed discrete subspace in the relative topology of Z_γ where γ is the supremum of the α involved, and so there exists $M \in \mathcal{M}_\gamma$ that meets $\{y_n : n \in \omega\}$ in an infinite set. In the former case, there trivially exists such an M in M_α.

In either case, there exists A_t, $t \in T_M$, almost contained in the set of all $y_n \in M$; so too $A_s \subset^* \{y_n : n \in \omega\} \cap M$ for every s immediately succeeding t in T_M; and so y_t is a limit point of $\{y_n : n \in \omega\}$.

Although it is not strictly needed for showing that Y has only trivial convergent sequences, it is convenient to show that Z_1 is locally countable and locally paracompact. Each basic neighborhood of $d_t \in D_0$ is the one-point compactification of a countable discrete space. This is also true of every weak basic neighborhood of $y_t \in Y_0$. The points of S_t are all on the same level of T, so their basic neighborhoods are almost disjoint. There thus exist basic nbhds, one for each point of S_t, that form a disjoint collection. The union of this collection, together with y_t, is an open nbhd of y_t that is homeomorphic to the well-known Arens space S_2; this is a regular space and, being countable, it is paracompact. No sequence of isolated points of U can converge to y_t: the corresponding fact about S_2 is well known. Since U is open, there is no sequence of distinct points of Y converging to y_t.

Let $1 < \alpha < \omega_1$ and suppose that, for all $\beta < \alpha$, no sequence of points in Y can converge to any point of Y_β. Each point of Y_α has a weak neighborhood W in Z homeomorphic to S_2, constructed in the same way as an actual neighborhood of a point of Y_0. List the relatively isolated points of W as $\{y_n : n \in \omega\}$. If $\alpha = \beta + 1$ then $y_n \in Y_b$ for all n, while if α is a limit ordinal, then $y_n \in \beta_n$ for some $\beta_n < \alpha$, and the β_n converge to α. To make an actual neighborhood for y, we attach neighborhoods to each y_n. Let V_n be an open neighborhood of y_n with all other points taken from Z_b [*resp.* Z_{β_n}]. Let $U = Y \cap \bigcup_{n\in\omega} V_n$. Then U is a Y-open neighborhood of y, and it is enough to show that no sequence from $U \setminus W$ converges to y.

The rest of the proof is a faint echo of the proof of [Ny2, Theorem 1.1]. Let $\langle p_n : n \in \omega \rangle$ be a 1-1 sequence of points of $U \setminus W$ with y in its closure. Since the sequence does not converge to y_0, there is an infinite $C_0 \subset \omega$ such that $\{p_n : n \in C_0\}$ does not have y_0 in its closure. Continue to inductively define $C_{k+1} \in [C_k]^\omega$ so that $\{p_n : n \in C_{k+1}\}$ does not have y_{k+1} in its closure. Finally, let C be an infinite set almost contained in C_k for all k. Then $\{p_n : n \in C\}$ does not have any y_n in its closure, so $U \setminus \{p_n : n \in C\}$ is a neighborhood of y witnessing that $\{p_n : n \in \omega\}$ does not converge to y. $\square$

Section 4. More about subsequential T_2 spaces

Now we will show the corollary of Theorem E mentioned in the introduction. Here they are again.

Theorem E. *Let Y be a subsequential T_2 space and let y be a nonisolated point of Y. If X is a countably compact T_2 space containing Y, then there is a nontrivial sequence in X converging to y.*

Corollary. *In a subsequential T_2 space, every countably compact subset is closed.*

Proof. Let S be T_2 and let Z be a countably compact subset of S. If $p \in \overline{Z} \setminus Z$, then $Y = Z \cup \{p\}$ is countably compact and p is nonisolated in Y but there is no sequence in Y converging to y. By Theorem E, Y cannot be embedded in a sequential space, and so neither can S. $\square$

In the terminology of [IN], this corollary says every subsequential T_2 space is C-closed:

4.1. Definition. A topological space is called *C-closed* [resp. *a KC-space*] iff every countably compact [*resp.* compact] subset is closed.

A well-known elementary fact is that every T_2 space is a KC-space, while every KC-space is clearly T_1. The property of being C-closed is much more restrictive. For instance, in [IN] it was shown that a sequentially compact T_2 space is sequential iff it is C-closed. The proof obviously extends to:

Theorem F. *A countably compact KC-space is sequential iff it is sequentially compact and C-closed.*

The following simple examples show that "KC-space" cannot be weakened to "T_1-space" nor even to "convergent sequences have unique limits."

4.2. Example. Let S_2 be the Arens space mentioned in the preceding section, with underlying set $\{x\} \cup (\omega \times (\omega + 1))$, with the product topology on $\omega \times (\omega + 1)$ and the cofinite subsets of $\{x\} \cup (\omega \times \{\omega\})$ containing x forming a weak base for the neighborhoods of x.

Let Z be the one-point compactification of $\omega \times (\omega + 1)$ with ∞ as the extra point. Let X be the quotient space of S_2 and Z formed by identifying the two copies of $\omega \times (\omega + 1)$. Convergent sequences in X have unique limits, and X is sequential, because sequentiality is preserved by quotient maps, and S_2 is sequential. Any infinite subset of X meets either the top row or $\omega \times \omega$ in an infinite set, and so either x or ∞ is an accumulation point, so X is countably compact (and countable, hence compact).

However, X is not a KC-space, and *a fortiori* not C-closed, because $(\omega \times \omega) \cup \{x\}$ is compact but not closed.

4.3. Example. This time, let Z be the one-point compactification of $\omega \times \omega$ with ∞ as the extra point. Let X be the quotient space of S_2 and Z formed by identifying the two copies of $\omega \times \omega$. This space has all the properties listed for the preceding one, except that each column is a sequence converging to two points, one of which is ∞. However, X is T_1.

Example 4.3 has a subspace relevant to the following rephrasing of Theorem F:

Theorem F′. *A countably compact, KC-space is sequential $\iff$ it is sequentially compact and every countably comact subset is compact.*

Here too, weakening "KC-space" to "convergent sequences have unique limits" results in a false statement, but in the opposite direction from Example 4.2, even for subsequential spaces.

4.4. Example. Let Y be the subspace of Example 4.3 obtained by removing $\omega \times \{\omega\}$. Convergent sequences in Y have unique limits, and Y is sequentially compact and countable, so every countably compact subset (including Y itself) is compact. Also, Y is subsequential since it is a subspace of Example 4.3, but it is not sequential because x is in the closure of $\omega \times \omega$ while every sequence in Y that converges to x is eventually constant.

The following problem, first posed at the 1980 Spring Topology Conference, remains unsolved:

Problem 2. Is every C-closed compact T_2 space sequential?

In any model where the answer to Problem 2 is Yes, so is the answer to Problem 1: see the Corollary. These models include those where MA or $2^\omega < 2^{\omega_1}$ [IN] holds. More generally:

Theorem G. [vD, Corollary 6.4] *If* $2^\omega < 2^{\mathfrak{t}}$ *then every C-closed compact* T_2 *space is sequentially compact, hence sequential.*

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