Discontinuities and smooth curves in n-space

I proved a strengthened converse to a theorem in our calculus text (Anton et al.) and would like to find out whether it is known already. The textbook theorem clearly extends to all $n \geq 2$:

Theorem 1. *If the limit of a real-valued function on $\mathbb{R}^2$ exists at a point p, then it will also be the limit along any smooth curve through p.*

The text also states the contrapositive in the following form: if the limit of f fails to exist on some smooth curve through p, or if f approaches different limits at p along two smooth curves, then the limit at p does not exist.

Definition 1. A *smooth curve* is the range of a C^∞ function $c : \mathbb{R} \to \mathbb{R}^n$ whose derivative is never the zero vector.

By the way, our calculus textbook defines "smooth" using C^1 instead of C^∞, but Theorem 1 and Theorem 2 (below) hold with either definition.

The part about $c'(t)$ never being $\mathbf{0}$ for a smooth curve is important. The existence of such a parametrization c is equivalent to the curve having a tangent line at each point. Without this restriction, we would be able to remove all hint of agreement between the limits along curves in the converse of Theorem 1. See Corollary 1 below.

Even as it is, we can confine ourselves to straight lines where disagreement is needed:

Theorem 2. *If f is a real-valued function defined in a deleted neighborhood of p in $\mathbb{R}^m$, and the limit of f at p does not exist, then either:*

(1) there is a smooth curve through p on which the limit does not exist, or

(2) there are two straight lines through p on which the limits exist, but are unequal.

Moreover, if (1) fails, and r_1 and r_2 are distinct limits along two straight lines, then every real number between r_1 and r_2 is the limit along some straight line through p.

Example 1. In Theorem 2, alternative (1) can indeed fail: consider the unit ball in $\mathbb{R}^3$, endowed with parallels of latitude like our earth, 0 at the equator and 90 at the poles (but no distinction between north and south, so that antipodes get the same value). Extend this real-valued function from the unit sphere inward, except to the origin $\mathbf{0}$, so that the resulting f is constant on each diameter (except for not being defined at the center $\mathbf{0}$).

It is easy to see that on any smooth curve through $\mathbf{0}$, this function approaches the value along the tangent line to the curve.

Corollary 1. *If f is a real-valued function defined in a deleted neighborhood of p in $\mathbb{R}^n$, and the limit of f at p does not exist, then there is a curve C through p which is the range of a C^∞ function, such that the limit of f does not exist on C.*

Proof. Splice together two rays that start at p on which f approaches different values, and parametrize the resulting angle so that we slow to zero velocity at p. $\square$

To prove Theorem 2, we begin with a definition which extends the concept of "tangent" from curves to sequences:

Definition 1. Let $p \in \mathbb{R}^m$ and let R be a ray starting at p. If $\langle p_n \rangle$ is a sequence converging to p, and a_n is the distance from p_n to p while b_n is the distance from p_n to R, we say $\langle p_n \rangle$ *is tangential to R at p* if $p_n \neq p$ for all n, but

$$\lim_{n\to\infty} \frac{b_n}{a_n} = 0.$$

This definition allows p_n to be on R itself.

Lemma 1. *If $p \in \mathbb{R}^n$ and $p_n \to p$, then there is a ray R starting at p and a subsequence of $\langle p_n \rangle$ that is tangential to R.*

Proof. By translation-invariance of the key concepts we may assume p is the origin $\mathbf{0}$. Let $\pi : \mathbb{R}^m \setminus \{\mathbf{0}\} \to S^{m-1}$ be the natural projection taking each open ray through the origin to its point of intersection with the unit sphere. Without loss of generality, we may assume the p_n are indexed so in one-to-one fashion so that no two are on the same ray.

Let $\langle p_{n_i} \rangle$ be a subsequence whose projection converges to a point $q \in S^{m-1}$. Let R be the ray from $\mathbf{0}$ to q. Then $\langle p_{n_i} \rangle$ is tangential to R. Indeed, if c_i is the distance from $\pi(p_{n_i})$ to R, then by similar triangles, $b_{n_i}/a_{n_i} = c_i \to 0$. $\square$

Lemma 2. *Let $\mathbf{0}$ be the origin in $\mathbb{R}^m$. We may assume without loss of generality that p is the origin $\mathbf{0}$. If $\langle p_n \rangle$ is tangential to a ray R at $\mathbf{0}$, then there is a C^∞ function $h : \mathbb{R} \to \mathbb{R}^m$ with nowhere zero derivative, whose range passes through infinitely many of the points p_n.*

Once Lemma 2 is proved, we can prove Theorem 2 as follows. Suppose first that f is unbounded in every neighborhood of $\mathbf{0}$. Then there is a sequence $\langle p_n \to \mathbf{0}$ such that $f(p_n)$ is monotone and unbounded. If h is as in Lemma 2, then its range is a smooth curve through $\mathbf{0}$ which witnesses alternative (1) of Theorem 2.

If f is bounded in some neighborhood of $\mathbf{0}$, then there are sequences $\langle p_n \rangle$ and $\langle q_n \rangle$ converging to p such that $\langle f(p_n) \rangle$ and $\langle f(q_n) \rangle$ converge to different numbers

r_1 and r_2. By Lemma 1 we may assume that $\langle p_n \rangle$ and $\langle q_n \rangle$ are tangent to rays R_1 and R_2, respectively. Let L_i be the line through p extending R_i. If $f \restriction L_i$ does not have a limit at p then we have alternative (1). If it has a limit x_i for both i but $r_i \neq x_i$ for some i, then connecting the range of h through p with the ray opposite R_i gives a smooth curve witnessing (1). Otherwise L_1 and L_2 witness (2).

Still assuming Lemma 2, we can prove the "moreover" part as follows. If (1) fails, then the limit of f exists along every line through p but we have lines L_1 and L_2 witnessing (2); let r_i be the limit along L_i. Let P be the plane determined by L_1 and L_2, and let R_i be either ray of L_i starting at p. Let A be a closed arc of the unit circle of P starting at R_1 and ending at R_2. For each ray R starting at p and passing through A, let $a(R)$ be the point in $R \cap A$.

Claim. If $\ell(R)$ is the limit at p of f along R, then the function g taking $a(R)$ to $\ell(R)$ is continuous.

Once the claim is proved, connectedness of A insures that every point between $r_1 = \ell(R_1)$ and $r_2 = \ell(R_2)$ is of the form $\ell(R)$ for some ray R that meets A, and Theorem 3 is proved.

Proof of Claim, still assuming Lemma 2. Suppose not, i.e., suppose we can have $a_n \to a$ but not $g(a_n) \to g(a)$. Let p_n be chosen on the ray through a_n so that $p_n \to p$. Then $\langle p_n \rangle$ is tangential to the ray through a, but some subsequence either is unbounded or approaches a number r_1 other than $g(a) = x_1$. Now argue as before to get a contradiction to the assumed failure of (1). This completes the proof of Theorem 2, modulo:

Proof of Lemma 2. By a symmetry argument, we may assume R is the positive x-axis $\mathbb{R}^+ \times \{0\} \cdots \times \{0\}$. Also, since differentiation goes coordinatewise, it is enough to prove Lemma 2 for $m = 2$. By taking a subspace if necessary, we may assume the second coordinate of $p_n = (x_n, y_n)$ is monotone, and so without loss of generality we may assume $\langle y_n \rangle$ strictly decreases to 0, the case where infinitely many y_n equal **0** being trivial.

There is a standard example of a C^∞ function $g : \mathbb{R} \to \mathbb{R}$ that is identically 0 on $(-\infty, 0]$, monotone, and identically 1 on $[1, \infty)$ [2, Example]. Similarly, (a, b) and (c, d), where $a < c$, can be joined by the graph of $h(x) = kg(x - a) + b$, where k is the slope of the line joining these two points. Since $\langle p_n \rangle$ is tangential to R, it is easy to inductively choose n_i so that if $g_i(x) = k_i g(x - x_{n_{i+1}}) + y_{n_{i+1}}$ joins $p_{n_{i+1}}$ to p_{n_i}, then k_i converges to 0 and hence so does the maximum value of each derivative of g_i as $i \to \infty$. Since all derivatives of g_iA are 0 at $x_{n_{i+1}}$ and x_{n_i}, a smooth curve results when we join the negative x-axis and the horizontal line $y = y_1$ to the union of the graphs of the g_i. With $f : \mathbb{R} \to \mathbb{R}$ the function whose graph this curve is, the parametrization $c(t) = (t, f(t))$ is C^∞, and the first coordinate of the tangent vector is 1 everywhere. $\square$