The new theorem in this paper (Theorem 2) negatively answers the following question of Gary Gruenhage:

Problem. *Is $C_k(\mathbb{Q})$ stratifiable?*

Here $C_k(X)$ stands for the set of continuous real-valued functions with domain X, with the compact-open topology. The question was motivated by the following theorem and conjecture of Gartside and Reznichenko:

Theorem 1. *$C_k(\mathbb{R} \setminus \mathbb{Q})$ is stratifiable.*

Conjecture. *Let X be a 0-dimensional separable metrizable space. Then $C_k(X)$ is stratifiable if, and only if, X is completely metrizable.*

The following theorem not only answers Problem 1 negatively, it also lends support to this conjecture.

Theorem 2. *Let X be a 0-dimensional separable metrizable space which is not scattered, and has the property that every compact subset is countable. Then $C_k(X)$ is not stratifiable.*

The proof of this theorem rests on the following theorem of Gartside and Reznichenko, which dispenses with the need to define either the compact-open topology or stratifiability.

Theorem 3. *Let X be a 0-dimensional separable metrizable space. Then $C_k(X)$ is stratifiable if, and only if, it is possible to assign to each clopen subset W of X a compact $F(W) \subset W$, and to each compact $K \subset X$ a compact $\phi(K) \supset K$ in such a way that, whenever $W \cap K \neq \emptyset$, it follows that $F(W) \cap \phi(K) \neq \emptyset$ also.*

For convenience, we say X has *the Gartside-Reznichenko property* if it has assignments $\phi(\cdot)$ and $F(\cdot)$ as above. We will show that if X satisfies the hypotheses of Theorem 2, then no pair of assignments $\{\phi(\cdot), F(\cdot)\}$ can witness the Gartside-Reznichenko property. Our strategy will be to define clopen sets W_n in X and a descending sequence of collections of compact sets $\mathcal{K}_n$ such that $W_n \cap \phi(K) = \emptyset$ for all $K \in \mathcal{K}_n$ but $W_n \cap K \neq \emptyset$ whenever $K \in \mathcal{K}_i$ for $i < n$, and such that $\bigcup_{n=0}^{\infty} W_n$ is clopen.

Once this is done, we need only set $W = \bigcup_{n=0}^{\infty} W_n$: since $F(W)$ is compact, $F(W) \subset \bigcup_{i=0}^{n} W_i$ for some n; then $W_{n+1} \cap K \neq \emptyset$ for some $K \in \mathcal{K}_n$, but $W_i \cap \phi(K) = \emptyset$ for $i \leq n$, so $W \cap K \neq \emptyset$ but $F(W) \cap \phi(K) = \emptyset$, and so X fails to have the Gartside-Reznichenko property.

To carry out this strategy, we introduce the following concept. Call a collection of countable (hence scattered) compact subsets of a metrizable space M *large* if it has members of arbitrarily high countable scattered height. Clearly every large collection. is uncountable, and if every countable subset of M is compact, the union of every large collection of compact sets has noncompact closure, since every countable compact space is scattered, and height does not increase in going to subspaces. The following is also obvious:

Lemma 1. *If a large collection is expressed as a union of countably many subcollections, at least one of the subcollections must also be large.* $\square$

Similarly, we have:

Lemma 2. *If $\mathcal{K}$ is large and $\{V_n : n \in \omega\}$ is a descending sequence of clopen sets whose intersection is finite, then there exists n such that $\{K \setminus V_n : K \in \mathcal{K}\}$ is large.*

Proof. If V_n is as above and K is compact and $\alpha_n \in \omega_1$ is an upper bound for the heights of the points in $K \setminus V_n$ then $sup_n \alpha_n + 1$ is an upper bound for the heights of the points in K. A proof by contrapositive is now immediate. $\square$

Lemma 3. *If M is a nowhere locally compact metric space, $\mathcal{K}$ is a large collection of countable compact subsets of M, $\phi(K)$ is a compact set for each $K \in \mathcal{K}$, and C is a nonempty clopen subset of M, then there is a nonempty clopen subset B of C such that $\{K \in \mathcal{K} : B \cap \phi(K) = \emptyset\}$ is large.*

Proof. Let $\{C_n : n \in \omega\}$ be a descending sequence of nonempty clopen subsets of C whose intersection is empty. By Lemma 1, all but finitely many C_n will do for B. $\square$

Proof of Theorem 2. By a well-known classical result, we may assume $X \subset \mathfrak{C}$ where $\mathfrak{C}$ stands for ${}^{\omega}2$, a.k.a. the Cantor set. For each finite sequence σ of 0's and 1's, let $B[\sigma]$ be the basic clopen subset of $\mathfrak{C}$ consisting of all points that extend σ. Let $\mathcal{B} = \{B[\sigma] : \sigma \in {}^{<\omega}2\}$. As is well known, $\mathcal{B}$ is a base for $\mathfrak{C}$, each member of which is homeomorphic to $\mathfrak{C}$ itself, with $B[\emptyset] = \mathfrak{C}$.

Lemma 4. *If $\mathcal{K}$ is large, and $\bigcup \mathcal{K} \subset B[\sigma]$, then there exists n such that there are at least two sequences σ_0, σ_1 of the same length extending σ such that $\mathcal{K} \restriction B[\sigma_i] = \{K \cap B[\sigma_i] : K \in \mathcal{K}\}$ is large for $i = 0, 1$.*

$\vdash$ *Proof of Lemma 4:* Let $\sigma \in {}^{n}2$ and let $m > n$. For each $K \in \mathcal{K}$, some point of maximal height in K is in one of the $B[\tau] (\tau \in {}^{m}2)$, so there is at least one $\tau \in {}^{m}2$ for which $\mathcal{K} \restriction B[\tau]$ is large. Suppose there is only one for each m. Then the associated clopen sets close down on a single point of $\mathfrak{C}$, and this contradicts Lemma 2. $\dashv$

For each compact $K \subset X$ let $\phi(K)$ be a compact subset of X. Define finite sequences σ and associated points $y_\sigma \in B[\sigma] \setminus X$, sets $B_\sigma \in \mathcal{B}$ such that $B_\sigma \subset B[\sigma]$, and large collections $\mathcal{K}_\sigma$ of compact sets by repeated application of Lemmas 1 through 4, in the following way. Begin with $\sigma = \emptyset$ and let $y_\emptyset$ be any point of $\mathfrak{C} \setminus X$. By Lemma 3, let $B_\emptyset$ be a neighborhood of $y_\emptyset$ such that $\{K \in \mathcal{K} : \phi(K) \cap B_\emptyset = \emptyset\}$ $(= \mathcal{K}_\emptyset)$ is large.

Suppose y_σ, etc. have been defined, in such a way that $\mathcal{K}_\sigma \restriction B[\sigma]$ is large, and $B_\sigma \cap \phi(K) = \emptyset$ for each $K \in \mathcal{K}_\sigma$, and B_σ is a neighborhood of y_σ in $B[\sigma]$. Applying

Lemma 4 to $B[\sigma]$, let σ_1 and σ_2 be distinct sequences of the same length, extending σ, for which $\mathcal{K}_\sigma \restriction B[\sigma_i]$ is large. Let y_{σ_i} be a point of $\overline{\bigcup \mathcal{K}_\sigma \cap B[\sigma_i]} \setminus X$ [overhead bars denote closure in $\mathfrak{C}$] and let B_{σ_i} be a neighborhood of y_{σ_i} in $B[\sigma_i]$ for which $\mathcal{K} = \{K \in \mathcal{K}_\sigma : K \cap B_{\sigma_i} = \emptyset\}$ is large. Let $\mathcal{K}_{\sigma_i} = \mathcal{K}$.

Once this induction is complete, the set of all σ for which y_σ, etc. have been defined is a copy of the full binary tree of height ω, and each branch defines a unique point of $\mathfrak{C}$. Moreover, each such point is in the closure of X, but not all of these points are in X, because the branches together define a copy of $\mathfrak{C}$.

Let y be one of these points in $\mathfrak{C} \setminus X$. The branch that runs to y defines a sequence of clopen subsets $B_\sigma \cap X$ of X. The union W of these sets is clopen since they converge on y. Re-index the B_σ and the $\mathcal{K}_\sigma$ by the natural numbers in order of the length of σ, and let $W_n = B_n \cap X$. These sets are exactly as required by the strategy explained above. $\square$

Title: Recent research on the compact-open topology and modifications

Let $C_k(X)$ stand for the space of continuous functions from X to $\mathbb{R}$ with the compact-open topology. For compact K, $C_k(K)$ is simply the Banach space given by the sup norm, but when X is not locally compact, $C_k(X)$ is very complicated. Gartside and Reznichenko [1] showed that $C_k(X)$ is stratifiable whenever X is a Polish space; as a result, $C_k(\mathbb{P})$ has emerged as a prime candidate for a negative solution to the 43-year-old problem of whether every stratifiable space is M_1. The following problem is also of interest:

Problem 1. *Let X be separable metrizable. If $C_k(X)$ is stratifiable, must X be completely metrizable?*

The converse is true [1]. Problem 1 easily reduces to the 0-dimensional case. Since every scattered metrizable space is completely metrizable, the only restriction on the following partial solution to Problem 1 is in the last clause in the hypothesis.

Theorem 1. [2] *Let X be a 0-dimensional separable metrizable space which is not scattered, and has the property that every compact subset is countable. Then $C_k(X)$ is not stratifiable.*

This result is new even in the special case $X = \mathbb{Q}$, answering a question posed by Gary Gruenhage at the 2003 Lubbock conference. Theorem 1 made use of the following elegant criterion in [1].

Theorem. *Let X be a 0-dimensional separable metrizable space. Then $C_k(X)$ is stratifiable if, and only if, it is possible to assign to each clopen subset W of X a compact $F(W) \subset W$, and to each compact $K \subset X$ a compact $\phi(K) \supset K$ in such a way that, whenever $W \cap K \neq \emptyset$, it follows that $F(W) \cap \phi(K) \neq \emptyset$ also.*

This theorem also figures in the proof of Theorem 2 below, which represents the first progress towards the solution of the following problem.

Problem 2. *Let $C_s(\mathbb{P}, \omega)$ stand for the set of continuous natural-number-valued functions on $\mathbb{P}$ with the sequential modification of the compact-open topology. Is $C_s(\mathbb{P}, \omega)$ 0-dimensional?*

The modification in question is the one in which a set is open iff it is sequentially open in $C_k(\mathbb{P}, \omega)$. Sequential convergence in $C_k(\mathbb{P}, \omega)$ has the following appealing characterization:

$$f_n \to f \iff f_n(x_n) \to f(x) \text{ whenever } x_i \to x.$$

Problem 2 may seem specialized, but a positive solution would be enough to solve a problem in theoretical computer science. This problem is whether two competing approaches to higher-type real-number computability actually coincide on level 3. References [3], [4] and [5] explain these concepts, and [5] shows how analogues of Problem 2, obtained by iterating the functor $C_s(\cdot, \omega)$, would establish the coincidence at all levels.

Definition. *A space X is* semiregular *if it has a base of regular open sets, and* countably 0-dimensional *if whenever $x \in X$ and F is a countable closed subset of X, then there is a clopen set containing x and missing F.*

Theorem 2. *$C_s(\mathbb{P}, \omega)$ is semiregular and countably 0-dimensional. In fact, if $x \in C_s(\mathbb{P}, \omega) \setminus F$ and F is a countable closed subset of $C_s(\mathbb{P}, \omega)$, then there is a set U that is open in $C_p(\mathbb{P}, \omega)$ and closed in $C_s(\mathbb{P}, \omega)$, contains F, and misses x.*

Here C_p refers to the product topology, which is much coarser than the compact-open topology in this context.

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