

WORKSHOP LECTURE ON PRODUCTS OF FRÉCHET SPACES

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ABSTRACT. The general question, “When is the product of Fréchet spaces Fréchet?” really depends on the questions of when a product of α_4 Fréchet spaces (also known as *strongly Fréchet* or *countably bisequential* spaces) is α_4 , and when it is Fréchet. Two subclasses of the class of strongly Fréchet spaces shed much light on these questions. These are the class of α_3 Fréchet spaces and its subclass of \aleph_0 -bisequential spaces. The latter is closed under countable products, the former not even under finite products. A number of fundamental results and open problems are recalled, some further highlighting the difference between being α_3 and Fréchet and being \aleph_0 -bisequential.

This paper is a slightly updated note for the first of two lectures presented by the author in his Workshop on Sequential Convergence at the ten-day Advances in Set-Theoretic Topology conference in Erice. Erice is a remarkable mountaintop town with a medieval feel to it, overlooking the northwestern tip of Sicily, and the author is grateful for the invitation to this unique conference.

Recall that a space is called **Fréchet (or: Fréchet-Urysohn)** if, whenever a point p is in the closure of a subset A , there is a sequence in A converging to p .

In this paper, “space” will mean “Hausdorff space,” although much of what we say holds for topological spaces in general.

This paper revolves around the following general problem, to which Tsugunori Nogura has made many basic contributions.

General Problem. When is the product of Fréchet spaces Fréchet?

1. Fundamental problems and theorems

The following space is very relevant to this general problem.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{T}\mathcal{E}\mathcal{X}$

Example 1. The **Fréchet fan**, here denoted F_ω , is the quotient of the space $\omega \times (\omega + 1)$ obtained by identifying all the nonisolated points to a single point p . Then the image of each copy $\{n\} \times \omega$ converges on p , and a subset has p in the closure if, and only if, it meets one of these images in an infinite set. A useful feature of F_ω is that it is homeomorphic to every subspace that contains p and meets infinitely many of these images in an infinite set.

Remark. In many papers, F_ω is denoted S_ω , but that symbol has also been used for a certain non-Fréchet sequential space.

It is easy to see that F_ω is Fréchet, yet its product with any nondiscrete Fréchet space fails to be Fréchet [M]! So we need some extra properties to get anywhere with the General Problem. Perhaps the most widely researched ones are the α_i -properties.

Definition 1. Let $i \in \{1, 1.5, 2, 3, 4\}$. A point p in a space X is an α_i -**point** if for each family $\{\sigma_i : i \in \omega\}$ of sequences with disjoint ranges converging to p , there is a sequence $\sigma \rightarrow p$ such that:

α_1 : $\text{ran}(\sigma_i) \subset^* \text{ran}(\sigma)$ for all i ; [$A \subset^* B$ means $A \setminus B$ is finite.]

$\alpha_{1.5}$: $\text{ran}(\sigma_i) \subset^* \text{ran}(\sigma)$ for infinitely many i ;

α_2 : $\text{ran}(\sigma_i) \cap \text{ran}(\sigma)$ is infinite (equivalently, nonempty) for all i ;

α_3 : $\text{ran}(\sigma_i) \cap \text{ran}(\sigma)$ is infinite for infinitely many i ;

α_4 : $\text{ran}(\sigma_i) \cap \text{ran}(\sigma)$ is nonempty for infinitely many i .

A space is an α_i -**space** if every point is an α_i -point.

First countable spaces and the one-point compactifications of discrete spaces are easy examples of α_1 Fréchet spaces.

Lemma 1. [Sw] [Ta] *A space [resp. regular space] is α_4 if, and only if, it does not contain a copy [resp. a closed copy] of F_ω .*

Corollary 1. *If a product of two nondiscrete Fréchet spaces is Fréchet, then both are α_4 .*

Theorem 1. [N2] *The product of countably many α_i -spaces is α_i for $i \in \{1, 2, 3\}$.*

In contrast, the general problem of when the product of α_4 spaces is α_4 is almost as intractable as the General Problem of this lecture. Example 2 below features a pair of compact Fréchet counterexamples which involve the following concepts:

Definition 2. An **AD family on ω** is a collection of infinite subsets of ω such that the intersection of any two is finite. An AD family is called **MAD** if it is an infinite maximal AD family. Maximality of \mathcal{A} means that every infinite subset of ω meets some member of \mathcal{A} in an infinite set.

[Some authors omit the first and/or second “infinite,” making the names true acronyms, but the usage adopted here saves space later on.]

Given an AD family \mathcal{A} on ω , the space $\Psi(\mathcal{A})$ is the locally compact space whose underlying set is the union of ω with a set of added points p_A ($A \in \mathcal{A}$) and where points of ω are isolated, while a neighborhood of p_A is any subset of $\Psi(\mathcal{A})$ which contains p_A and all but finitely many points of A . Let $\Psi^*(\mathcal{A})$ denote the one-point compactification of $\Psi(\mathcal{A})$.

Example 2. Simon [S1] showed that there is a MAD family \mathcal{M} on ω which is the union of two subcollections \mathcal{A}_0 and \mathcal{A}_1 neither of which traces a MAD family on any subset of ω . As a result, both $\Psi^*(\mathcal{A}_i)$ are Fréchet but their product is not Fréchet.

Nogura [N2] deleted denumerably many members of \mathcal{M} and showed that the resulting $\Psi^*(\mathcal{A}'_i)$ do not have α_4 (nor Fréchet) product.

Remarkably enough, the following question was not fully answered until this year.

Question 1. *Is there a ZFC example of a pair of α_4 Fréchet spaces, neither of which is α_3 , whose product is Fréchet?*

A CH example was provided back in 1987 by Nogura [N3]; see Example 4 below. The ZFC answer by Petr Simon [who was apparently unaware of Nogura's CH solution], appears in [S3] and takes the form of a single space with the property that all of its finite powers are Fréchet (and hence also α_4 , see Corollary 1).

Turning now to positive results, Arhangel'skiĭ showed:

Theorem 2. [A] *If X is an α_3 Fréchet space, then $X \times Y$ is Fréchet for every regular countably compact Fréchet space.*

Problem 1. *Is the converse true? No if CH: Nogura [N3].*

We will return to this problem later. Here is an immediate consequence of Corollary 1 and Theorem 2:

Corollary 2. *Every regular, countably compact Fréchet space is α_4 .*

Theorem 3. [N2] *If Y is a countably compact regular space and $X \times Y$ is Fréchet, then $X \times Y$ is also α_4 .*

Nogura also showed:

Theorem 4. [N2] *The class of countably compact regular α_i Fréchet spaces is countably productive for $i \in \{1, 2, 3\}$.*

Another countably productive class of α_3 Fréchet spaces is the class of \aleph_0 -bisequential spaces.

Definition 3. A space X is **bisequential** if, whenever \mathcal{U} is an ultrafilter converging to $p \in X$, there is a countable filterbase $\mathcal{F} \subset \mathcal{U}$ which also converges to p .

A space X is \aleph_0 -**bisequential** if every countable subset of X is bisequential and, for some (equivalently, every) compactification bX of X , if $x \in X$ and $x \in cl_{bX}C$ and

$$C \subset \bigcup \{cl_{bX}B : B \subset X, |B| \leq \aleph_0\}$$

then there exists a countable subset D of C such that $x \in cl_{bX}D$.

In particular, a compact space is \aleph_0 -bisequential iff it is Fréchet and every countable subset is bisequential. Also [A] a separable space is \aleph_0 -bisequential iff it is bisequential.

Theorem 5. [A] *The class of \aleph_0 -bisequential spaces is closed under countable products, and the product of an \aleph_0 -bisequential space and an α_4 Fréchet space is both α_4 and Fréchet.*

The resemblance between Theorem 5 on the one hand, and Theorems 2 and 4 on the other hand may not be completely accidental:

Problem 2. *Is every compact [resp. countably compact regular] α_3 Fréchet space \aleph_0 -bisequential?*

I am even unaware of any consistent counterexamples, although Example 5 below holds out hope (see Problem 4 at the end). The class of \aleph_0 -bisequential spaces is hereditary, so a Yes answer to Problem 2 would give a proper containment. A negative answer to Problem 2 would give an affirmative one to the following problem, in view of Theorem 4:

Problem 1⁺. *Is there a ZFC example of a space X that has Fréchet product with every regular countably compact Fréchet space, but is not \aleph_0 -bisequential?*

In [JM], Jordan and Mynard characterized those Fréchet spaces whose product with every α_4 Fréchet space is Fréchet. Naturally enough, they called these spaces **productively Fréchet**. By Theorem 5, \aleph_0 -bisequential spaces are productively Fréchet. Is it consistent that the converse holds? To put it another way:

Problem 3. *Is there a ZFC example of a space that is productively Fréchet, but is (a) not α_3 ? (b) not \aleph_0 -bisequential?*

Example 5 below is a consistent example for part (a). A ZFC example for Problem 3(b) would also be one for Problem 1⁺, by Corollary 2.

Theorem 6. [JM] *If X is productively Fréchet, then $X \times Y$ is also α_4 for every α_4 Fréchet space Y .*

Theorems 3, 5, and 6 suggest the following question, already asked by Nogura in [N2].

Question 2. *If the product of two nondiscrete spaces is Fréchet, must it also be α_4 ? [By Corollary 1, the factors themselves are α_4 .]*

This turns out to be ZFC-independent. On the one hand, Petr Simon showed [S2] that CH implies that the answer is negative; on the other hand, Todorčević showed [T] that the answer is affirmative under the Open Coloring Axiom (OCA).

Nogura also asked the “dual” to Question 2 in [N2]:

Question 3. *If the product of two Fréchet spaces is α_4 , must it be Fréchet?*

Fifteen years later, a ZFC counterexample was found by Costantini and Simon [CS]. Earlier, Costantini constructed an example assuming MA [C].

2. Applications of \mathbb{N}^* and examples

Many counterexamples in the theory of Fréchet spaces are denumerable sets with a single nonisolated point. After all, if a space X fails to be Fréchet or α_i , there is a subspace of this form where the property also fails. We adopt the notation $X = \mathbb{N} \cup \{\infty_X\}$ (or $\mathbb{N} \cup \{\infty\}$) for these spaces.

The topology of the Stone-Čech remainder \mathbb{N}^* sheds a great deal of light on these spaces. Recall that if $A \subset \mathbb{N}$, then the remainder $A^* = \mathcal{C}\ell_{\beta\mathbb{N}}A \setminus A$ is a clopen subset of \mathbb{N}^* , and every clopen subset of \mathbb{N}^* is of this form. If \mathcal{F} is the neighborhood filter of ∞ then the closed subset $\mathcal{F}_* = \bigcap \{F^* : F \in \mathcal{F}\}$ of ω^* determines \mathcal{F} and vice versa. Thus we also use the notation $\mathbb{N} \cup \{\mathcal{F}_*\}$ for $\mathbb{N} \cup \{\infty\}$. This is motivated by the fact that $\mathbb{N} \cup \{\infty\}$ is the quotient space of $\mathbb{N} \cup \mathcal{F}_*$ obtained by identifying \mathcal{F}_* to a single point.

Here are some key facts about this relationship.

Fact 1. If $A, B \subset \mathbb{N}$, then $A^* \subset B^*$ iff $A \subset^* B$, where $A \subset^* B$ means $A \setminus B$ is finite. Hence $A^* \cap B^* = \emptyset$ if and only if $A \cap B$ is finite.

Fact 2. A sequence σ in \mathbb{N} converges to ∞ iff $\sigma^* \subset \mathcal{F}_*$. [I use σ^* as shorthand for $(\text{ran } \sigma)^*$.] Also, σ clusters at ∞ iff σ^* meets \mathcal{F}_* .

A consequence is Malyhin’s 1972 observation:

Fact 3. $\mathbb{N} \cup \{\mathcal{F}_*\}$ is Fréchet iff \mathcal{F}_* is regular closed.

Fact 4. $\mathbb{N} \cup \{\mathcal{F}_*\}$ is a copy of F_ω iff \mathcal{F}_* is the closure of a non-clopen cozero set of \mathbb{N}^* .

Fact 5. [N2] The subspace $\Delta \cup \{(\infty_X, \infty_Y)\}$ of the product of $X = \mathbb{N} \cup \mathcal{F}_*$ and $Y = \mathbb{N} \cup \mathcal{G}_*$ is homeomorphic to $\mathbb{N} \cup \{\mathcal{F}_* \cap \mathcal{G}_*\}$. [Here Δ denotes $\{(n, n) : n \in \mathbb{N}\}$.]

Now, the intersection of two regular closed sets does not have to be regular closed; so Fact 5 has been used to concoct many pairs of Fréchet spaces whose product is not Fréchet. One of the first was:

Example 3. [O] An (ω_1, ω_1) -gap in \mathbb{N} is a pair $(\mathcal{A}, \mathcal{B})$ of ω_1 -sequences of subsets of \mathbb{N} such that

- (1) $A_\eta \subset^* A_\xi$ and $B_\eta \subset^* B_\xi$ whenever $\eta < \xi$ and
- (2) $A_\alpha \cap B_\beta$ is finite for all $\alpha, \beta < \omega_1$ and
- (3) if $A_\alpha \subset^* A$ for all α then $A \cap B_\beta$ is infinite for some (equivalently, cofinally many) β .

Now, if \mathcal{A} is a \subset^* -ascending ω_1 -sequence of subsets of \mathbb{N} , then $\mathcal{A}^* = \bigcup\{A_\alpha^* : \alpha \in \omega_1\}$ is what I call a ω_1 -**oval**: a union of a chain of clopen subsets of \mathbb{N} of cofinality ω_1 . And if $(\mathcal{A}, \mathcal{B})$ is an (ω_1, ω_1) -gap, then \mathcal{A}^* and \mathcal{B}^* are disjoint open sets whose closures meet. Indeed, every ultrafilter extending the (proper) filterbase

$$\mathcal{S} = \{L \setminus M : \forall \alpha \in \omega_1 (A_\alpha \subset^* L \wedge |B_\alpha \cap M| < \omega)\}$$

is in $cl_{\mathbb{N}^*}\mathcal{A}^* \cap cl_{\mathbb{N}^*}\mathcal{B}^*$. (Conversely, if $(\mathcal{A}, \mathcal{B})$ is a pair of \subset^* -ascending ω_1 -sequences of subsets of \mathbb{N} such that \mathcal{A}^* and \mathcal{B}^* are disjoint open sets whose closures meet, then $(\mathcal{A}, \mathcal{B})$ must also satisfy conditions (2) and (3) above, and hence is an (ω_1, ω_1) -gap.)

On the other hand, the closures of \mathcal{A}^* and \mathcal{B}^* (which are regular closed by definition) meet only on their boundaries, hence their intersection is not regular closed — in fact, it is nowhere dense. So by Fact 5, the product of the Fréchet spaces $X = \mathbb{N} \cup \{cl_{\mathbb{N}^*}\mathcal{A}^*\}$ and $Y = \mathbb{N} \cup \{cl_{\mathbb{N}^*}\mathcal{B}^*\}$ is not Fréchet.

Nevertheless, both X and Y are α_2 , simply because both \mathcal{F}_* and \mathcal{G}_* are the closures of ω_1 -ovals. Indeed, if σ_n converges to ∞_X for each n , then σ_n^* meets \mathcal{A}^* and hence some $A_{\alpha_n}^*$; so if $\beta = \sup_n(\alpha_n)$ and σ lists A_β then the range of each σ_n has an infinite intersection with the range of σ .

I like to call the following example “Nogura’s Lakes of Wada,” after the 1917 example of Kunizo Yoneyama, attributed to his teacher Takeo Wada, of three regions (“lakes”) in the plane which share a common boundary.

Example 4. Nogura [N2], using CH, designed a pair of disjoint ω_1 -ovals U and V in \mathbb{N}^* whose closures meet as in Example 3, with the additional properties that

- (4) $\mathbb{N}^* \setminus (U \cup V)$ is the closure of a cozero set C and
- (5) $Fr(U) = Fr(V) = Fr(C)$ [Fr stands for frontier, or boundary].

It follows that $U \cup \overline{C} = \overline{U \cup C}$ and $V \cup \overline{C} = \overline{V \cup C}$. Thus $\overline{U \cup C} \cap \overline{V \cup C} = \overline{C}$ and so $\mathbb{N} \cup \{\overline{C}\}$ is a copy of F_ω . Hence, by Lemma 1 and Fact 5, if $X = \mathbb{N} \cup \{\overline{U \cup C}\}$ and $Y = \mathbb{N} \cup \{\overline{V \cup C}\}$ then $X \times Y$ is not α_4 .

On the other hand, (5) guarantees [N2] [N3] that X and Y are both α_4 , and both are Fréchet by Fact 2. By Theorem 1, they cannot both be α_3 , and Nogura showed [N3, in effect] that neither is α_3 . Nogura did not know whether $X \times Y$ is Fréchet but he did show that $X \times K$ is Fréchet for every countably compact regular Fréchet space K [N3]. Thus if Z is one of Petr Simon’s factor spaces in Example 2,

then $X \times Z$ is Fréchet but neither factor is α_3 . This shows that CH gives consistent negative answers to Question 1 and Problem 1, and accounts for the “ZFC” in Problem 1⁺. Here is a more perspicacious variation on Problem 1:

Problem 1’. *Characterize those Fréchet spaces X whose product with every regular countably compact (or compact) Fréchet space is Fréchet. α_3 is sufficient for X , by Theorem 2; is it consistent that it is necessary?*

If we weaken “regular countably compact” to “ α_4 ” then α_3 is no longer sufficient for X . Example 3 even gave an α_2 Fréchet space that is not productively Fréchet. On the other hand, the following example shows that even here, it is consistent that α_3 is not necessary either.

Notation. *If f and g are functions from ω to ω , we write $f <^* g$ to mean that the graph of g is eventually above the graph of f ; in other words, $f(n) < g(n)$ for all but finitely many $n \in \omega$.*

Example 5. Let $\langle f_\alpha : \alpha < \mathfrak{b} \rangle$ be a $<^*$ -unbounded, $<^*$ -well-ordered family of increasing functions from ω to ω . Let $X = \Psi(\omega \times \omega, \mathcal{A})$ where now $\omega \times \omega$ is the dense set of isolated points and A_α is the graph of f_α and $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{b}\}$.

Then $X + \infty$, the one-point compactification of X , is Fréchet [Ny], and in fact, productively Fréchet [JM]. But it is consistent that it not be α_3 . The columns converge to the extra point ∞ , and if we list all subsets of $\omega \times \omega$ that meet infinitely many columns in an infinite set as $\{S_\alpha : \alpha < \mathfrak{c}\}$, then the axiom $\mathfrak{b} = \mathfrak{c}$ lets us define f_α so that its graph meets S_α in an infinite set. Thus no sequence with range S_α can converge to ∞ , because $\{\infty\} \cup (X \setminus A_\alpha)$ is a neighborhood of ∞ missing an infinite subset of S_α . Hence if $\mathfrak{b} = \mathfrak{c}$ then X can be constructed so as not to be α_3 .

A similar argument works in the following setting. In any model $V[G]$ constructed by iterated ccc forcing, Cohen reals are added at limit stages. So if such a forcing is of cofinality $\mathfrak{b}^{V[G]}$, we can look at the final model, and define f_α by induction at limit α so that its graph is not only eventually above the graph of every f_β such that $\beta < \alpha$, but also has infinite intersection with every set in $V[G_\beta]$ that meets infinitely many columns in an infinite set. Since every countable subset of $\omega \times \omega$ in $V[G]$ occurs in some initial model, the same argument as with $\mathfrak{b} = \mathfrak{c}$ shows that X is not α_3 .

In contrast, Michael Hrušak has shown that it is consistent for Example 5 to be bisquential no matter how the family of f_α s is defined. We also have:

Theorem 7. *There is a ZFC version of Example 5 that is bisquential.*

Proof. We do the construction so that $X \setminus \{\infty\}$ has a coarser separable metrizable topology. There are standard arguments that this implies X is bisquential, but we will give a direct proof of bisquentiality below.

Make $f_\alpha(n)$ be squarefree and have exactly $n+1$ prime divisors, and have $f_\alpha(n)$ divide $f_\alpha(n+1)$ for all α and n , with the quotient being a bigger prime than any that divide $f_\alpha(n)$.

This works! Let \mathcal{P}_k be the partition of $\omega \times \omega$ in which the first member is all points whose 2nd coordinate is 0 or 1, whose second member is all points whose 2nd coordinate has fewer than k prime divisors (counting repetitions), and whose $j+2$ nd member is those (x, y) such that y has the j th prime as its k th smallest prime divisor (again counting repetitions). Note that the graph of every f_α is almost contained in exactly one member of each partition. Also, if $\alpha \neq \beta$ then there is k such that f_α and f_β (or, more precisely, their graphs A_α and A_β) are almost contained in different members of \mathcal{P}_k .

Now if we let \mathfrak{Q} be the collection of all partitions of $\omega \times \omega$ into finitely many pieces, all but one of which is a singleton, then $\mathfrak{Q} \cup \{\mathcal{P}_k : k \in \omega\}$ is a countable subbase for a separated uniformity on $\omega \times \omega$ which extends to a countable subbase \mathfrak{R} for a uniformity on $X \setminus \{\infty\}$ in the natural way: by the arguments in the preceding paragraph, the closures of the members of each partition are disjoint, and their union covers $X \setminus \{\infty\}$. This gives the coarser metrizable uniformity mentioned above, but we will not need this information below.

Let \mathcal{U} be an ultrafilter on X . If \mathcal{U} is fixed on some point x then letting $\{\{x\}\} = \mathcal{F}$ in the definition of “bisequential” obviously works for \mathcal{U} . Otherwise, \mathcal{U} “lives” on $X \setminus \{\infty\}$ and contains every cofinite subset of X .

Next assume $\omega \times \omega \notin \mathcal{U}$; in other words, \mathcal{U} is a free ultrafilter which “lives” on the set $M = \{A_\alpha : \alpha \in \mathfrak{b}\}$ and thus converges to ∞ . Let us associate to every $\varphi \in {}^{<\omega}2$ a subset M_φ of M in such a way that:

- (1) $M_\emptyset = M$ and, for every $n \in \omega$, $\{M_\varphi : \varphi \in {}^n 2\}$ is a (faithfully indexed) partition of M ;
- (2) If ψ extends φ then $M_\psi \subset M_\varphi$;
- (3) $\forall f \in {}^n 2 \quad |\bigcap_{n \in \omega} M_{f \upharpoonright n}| \leq 1$.

(Clearly, such an association does exist for every set M of cardinality no greater than \mathfrak{c} .) Then, since \mathcal{U} is an ultrafilter, there is a unique $g \in {}^n 2$ such that for every $n \in \omega$, $M_{g \upharpoonright n}$ is the unique element of $\{M_\varphi : \varphi \in {}^n 2\}$ which belongs to \mathcal{U} .

Notice that, by (3), the set $L = \bigcap_{n \in \omega} M_{g \upharpoonright n}$ contains at most one element; and, since \mathcal{U} is free, $M \setminus L \in \mathcal{U}$. It follows that $\{M_{g \upharpoonright n} \setminus L : n \in \omega\}$ is a countable filterbase included in \mathcal{U} which converges to ∞ . Specifically, if $V = \{\infty\} \cup (M \setminus F)$ is a neighborhood of ∞ in $M \cup \{\infty\}$, then F is finite and so $M_{g \upharpoonright n} \setminus L \subset F$ for sufficiently large n .

Finally, if $\omega \times \omega \in \mathcal{U}$, then either each \mathcal{P}_k contains exactly one member that belongs to \mathcal{U} , or else there exists n such that \mathcal{U} contains the complement of each member of \mathcal{P}_n . In either case, there is a descending ω -sequence of members of \mathcal{U}

such that there is at most one f_α whose graph is not almost disjoint from all but finitely many members of the sequence. If there is no such f_α , the filter whose base is the descending sequence and all cofinite members thereof converges to ∞ . If there is one, the ultrafilter either includes the graph of f_α , in which case it converges to p_{f_α} and so does the trace of the descending sequence on the graph of f_α , or else we can subtract off the graph and proceed as in the case where there is none. \square

Problem 4. *Is it consistent for there to be a version of Example 5 that is α_3 without being bisequential (equivalently, \aleph_0 -bisequential)?*

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