CLOSED 2-1 PREIMAGES OF ω_1 AND COHERENT 2-COLORING SYSTEMS

PETER NYIKOS

Abstract. [To be decided later]

1. Introduction

This paper was a long time in the writing. Sections 7 and 8 go all the way back to 1980, with them essentially completed at the SETOP conference in Missisagua, Ontario. Most of Section 3 was completed in 1986 with Fremlin's PFA breakthrough [4]; the remainder came in 2003, along with Theorems 6.1 and 6.2 in Section 6. Sections 2 and 9 go back to 1997, shortly after seeing [1] and Eisworth's proof [3] of the consistency of CC_{22} [see section 9 for the statement].

Conventions: "space" means "Hausdorff space," and "closed," when applied to functions, includes continuity. The symbol Λ will stand for the set of countable limit ordinals, and these are not taken to include 0. We let Λ_2 stand for the derived set of Λ; in other words, Λ_2 is the set of limits of limit ordinals in ω_1. Thus $\Lambda \setminus \Lambda_2$ is the set of all countable ordinals of the form $\beta + \omega$. We use T_5 to mean "hereditarily normal."

For ordered pairs of ordinals we use the notation $\langle , \rangle$ and reserve parentheses for intervals of ω_1 and for ordered pairs in sets where there is no linear order. When f is a function whose domain consists of ordered pairs we follow the custom of writing $f(a, b)$ for $f(\langle a, b\rangle)$ and $f((a, b))$.

2. Coherent 2-coloring systems: the basics

Given a set X, a subset S of X, and a collection $\mathcal{A}$ of subsets of X, we let $\mathcal{A} \restriction S = \{A \cap S : A \in \mathcal{A}\}$ and call it the *restriction* of $\mathcal{A}$ to S.

Definition 2.1. Let X be a subset of ω_1 and let $\mathcal{A}$ be a collection of subsets of X. A *coherent 2-coloring system on* $\mathcal{A}$ is a collection $\mathcal{C} = \{C_A : A \in \mathcal{A}\}$ such that C_A is a 2-coloring of A such that if $B \in \mathcal{A}$ then C_A agrees with C_B on all but finitely many $x \in A \cap B$.

We will use the words "black" and "white" in referring to the two colors, and let W_A and B_A be the set of points colored white and black, respectively, by C_A.

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The usual convention is that each ordinal is the set of all smaller ordinals. So a *2-coloring system on* ω_1 is a collection $\mathcal{C} = \{C_\alpha : \alpha \in \omega_1\}$ where each C_α is a 2-coloring of $\alpha = [0, \alpha)$. More generally, if X is a subset of ω_1 and $\mathcal{A} = \{\alpha \cap X : \alpha \in X\}$ then we we will also call $\mathcal{C}$ *a 2-coloring system on* X.

Definition 2.2. Given a subset Z of X, a coherent 2-coloring system on X is said to be *uniformizable on* Z if there is a 2-coloring of Z which agrees with each 2-coloring C_α in the system on all but finitely many $\xi \in Z \cap \alpha$.

If we know any of C_A, W_A or B_A, we can immediately reconstruct the other two if we know A. So the theory of coherent 2-coloring systems on ω_1 is equivalent to that of what is called coherent sequences on ω_1 in [1]:

Definition 2.3. A *coherent sequence on* ω_1 is a transfinite sequence $\mathcal{W} = \{W_\alpha : \alpha \in \omega_1\}$ such that $A_\alpha \cap \beta =^* A_\beta$ for all α, β in ω_1, where $X =^* Y$ means that the symmetric difference $X \Delta Y$ is finite.

Uniformizable 2-colorings correspond to what are called trivial sequences in [1]:

Definition 2.4. A coherent sequence is *trivial* if there is a set $A \subset \omega_1$ such that $A_\alpha =^* A \cap \alpha$ for all $\alpha < \omega_1$.

The existence of nontrivial coherent sequences (hence of non-uniformizable coherent 2-coloring systems) is deducible from the ZFC axioms: see [8] or Example 8.1. On the other hand, it is ZFC-independent whether there is a coherent 2-coloring system on ω_1 which is not uniformizable on any uncountable subset of ω_1: See Theorem 2.7 and Theorem 9.6 or [1].

The following concept will be used in constructing the nontrivial examples in this paper.

Definition 2.5. Let S be a subset of ω_1. A *ladder system on* S is a family $\mathcal{L} = \{L_\alpha : \alpha \in S \cap \Lambda\}$ of subsets of ω_1, of order type ω, such that each L_α, called *the ladder at* α, has supremum α.

Where there is no danger of confusion, we let each L_α be listed in its natural order as $\{\alpha_n : n \in \omega\}$, as in the following construction:

Construction 2.6. Let $\mathcal{L}$ be a ladder system on ω_1. Since the intersection of any two ladders is finite, every 2-coloring system on $\mathcal{L}$ is coherent. Moreover, it can be extended recursively to a coherent 2-coloring system $\mathcal{C}$ on ω_1 in the following way. With each L_α listed as $\{\alpha_n : n \in \omega\}$, let $\alpha_{-1} = 0$. If n is a finite ordinal, let $W_n = \emptyset, B_n = n$. If ξ is an infinite successor ordinal, $\xi = \eta + 1$, let λ be the greatest limit ordinal $< \xi$, and let $W_\xi = W_\lambda$. Finally, if $\alpha \in \Lambda$, let $W_\alpha \cap (\alpha_n, \alpha_{n+1}) = W_{\alpha_{n+1}} \cap (\alpha_n, \alpha_{n+1})$ for all $n \in \omega \cup \{-1\}$, and let $\alpha_n \in W_\alpha$ iff it is colored white by the 2-coloring of $L_\alpha \in \mathcal{L}$. It is easy to show by transfinite induction that this is a coherent 2-coloring.

Here is a simple but striking application of this construction. The axiom $\clubsuit$ states that there is a ladder system $\mathcal{L} = \{L_\alpha : \alpha \in \Lambda\}$ such that for every uncountable subset E of ω_1, there exists α such that $L_\alpha \subset E$.

Theorem 2.7. *If* $\clubsuit$*, there is a coherent 2-coloring on* ω_1 *that is not uniformizable on any uncountable set.*

Proof. Let $\mathcal{L}$ be a ladder system witnessing $\clubsuit$. With notation as in the foregoing construction, let $\alpha_{2n} \in W_\alpha$ and $\alpha_{2n+1} \in B_\alpha$ for all $n \in \omega$ and all $\alpha \in \omega_1$, and extend this 2-coloring on $\mathcal{L}$ to one on ω_1 as above. If E is an uncountable subset of ω_1, let C_E be any 2-coloring of E and let $L_\alpha \subset W_E$. Then C_E disagrees with C_α on infinitely many elements of α, and so it does not uniformize this 2-coloring on E. □

3. Closed 2-1 preimages of ω_1: general facts

Two-to-one closed preimages of ω_1 have a pleasingly simple structure that makes many nice pictures possible, but they also have played a big role in research. The 1986 proof by D. H. Fremlin that it is consistent that they all contain copies of ω_1 opened the gates to a flood of research that culminated in Balogh's solution of the Moore-Mrówka problem, and Balogh's theorem that it is consistent that every first countable countably compact space is either compact or contains a copy of ω_1.

I refer to 2-1 closed preimages of ω_1 as "sprats" after the nursery rhyme about Jack Sprat and his wife [which, like many nursery rhymes, had an origin in political protest]. This is because at each limit ordinal, whatever one point's neighborhoods don't gobble up beyond some earlier stage, the other point's neighborhoods will.

To put that last bit more formally ... every 2-1 preimage of ω_1, closed or otherwise, can be given $\omega_1 \times 2$ as an underlying set, where as usual $2 = \{0, 1\}$. The map to ω_1 is thought of as a projection and labeled π even if (as is usually the case) the topology on $\omega_1 \times 2$ is *not* the product topology.

Now, if α is a limit ordinal and we pick disjoint open nbhds U_0 and U_1 respectively of $\langle\alpha, 0\rangle$ and $\langle\alpha, 1\rangle$, then closedness of π implies that there is no sequence from the complement of $U_0 \cup U_1$ whose projection converges up to α; otherwise the sequence would be closed in the domain; but its projection is not closed in ω_1. Thus there is $\xi < \alpha$ such that $\pi^{\leftarrow}(\xi, \alpha] \subset U_0 \cup U_1$. By chopping off U_0 and U_1 at ξ we thus get a pair of disjoint basic clopen nbhds of $\langle\alpha, 0\rangle$ and $\langle\alpha, 1\rangle$. Extending these back and using induction and the fact that ω_1 is well-ordered, we can define a partition of $\pi^{\leftarrow}[0, \alpha]$ into two disjoint clopen sets $B(\alpha, 0)$ and $B(\alpha, 1)$. Of course it is enough to define $B(\alpha, i)$ for one i.

The above argument also shows $\pi^{\leftarrow}[0, \alpha]$ is countably compact. Since it is also countable, it is compact and hence first countable. In fact sets of the form $B(\alpha, i) \setminus \pi^{\leftarrow}[0, \beta]$ give a base of compact clopen nbhds at $B(\alpha, i)$ as β ranges over the countably many ordinals less than α.

By the conventions, the two points above 0 and the two points above any successor ordinal are isolated. Thus every sprat is locally compact and countably compact.

Example 3.1. The product space $\omega_1 \times 2$ is a sprat. We have $B(\alpha, i) = [0, \alpha] \times \{i\}$ for all limit α.

Example 3.2. If we let $B(\alpha, 1) = \{\langle\alpha, 1\rangle\}$ for all $\alpha \in \omega_1$, then we have the Alexandroff duplicate of ω_1 in which all the points of $\omega_1 \times \{1\}$ are isolated.

It is not hard to show that this sprat is homeomorphic to ω_1. In fact the following will be useful here and later:

Classical Theorem. *Every countable, compact space is homeomorphic to a countable ordinal.*

Folklore Theorem. *A space is homeomorphic to ω_1 iff it is the union of a strictly ascending sequence of countable compact open sets $\langle K_\xi : \xi < \omega_1\rangle$ such that $K_\beta \setminus \bigcup\{K_\xi : \xi < \beta\}$ is a singleton for a club set of β's.*

Example 3.3. For each limit ordinal α let $\langle \alpha_n : n \in \omega\rangle$ be an increasing sequence of successor ordinals converging up to α. Let

$$B(\alpha, 1) = \{\langle \alpha, 1\rangle\} \cup \{\langle \alpha_n, 0\rangle : n \in \omega\}.$$

Unlike the first two examples, this sprat is not hereditarily collectionwise Hausdorff: $\Lambda \times \{0\}$ is a closed copy of ω_1 and if we remove it, the points of $\Lambda \times \{1\}$ comprise a closed discrete subset of what remains, which cannot be expanded to a disjoint collection of open sets because of the Pressing Down Lemma.

Despite its seeming simplicity, Example 3.3 is sufficiently complicated that it is ZFC-independent whether it is hereditarily normal (T_5); see Section 6.

Since π is a closed map, every uncountable closed subset of a sprat has a club subset of ω_1 as an image. The pigeonhole principle then gives:

Lemma 3.4. If a sprat has two disjoint uncountable closed sets, then there is a club $C \subset \omega_1$ such that both sets meet each fiber over C exactly once.

If F_0 and F_1 are a pair as in Lemma 3.4, and $\Omega_i = \pi^{\leftarrow} C \cap F_i$ then Ω_0 and Ω_1 are disjoint closed copies of ω_1. This is because the restrictions of π to Ω_0 and Ω_1 are one-to-one, continuous, and closed, and because every club subset of ω_1 is a homeomorphic copy of ω_1. This gives us:

Corollary 3.5. If a sprat does not have two disjoint closed copies of ω_1, then it is normal.

Thus Examples 3.2 and 3.3 are normal, and Example 1 is obviously normal also. But it is unusual in one way:

Theorem 3.6. *If a sprat* **does** *have two disjoint copies of ω_1, then it is normal $\iff$ the copies can be put into disjoint open sets $\iff$ it is homeomorphic to $\omega_1 \times 2$.*

Only the last $\Longrightarrow$ requires work, and the Folklore Theorem does most of it for us.

4. More connections

In this section we study some connections between 2-colorings, ladders, sprats, and two other important concepts: Aronszajn trees and the Stone-Čech remainder of the discrete space of cardinality $\aleph_1$.

We begin with a simple construction which produces a sprat with two disjoint copies of ω_1 from a given coherent 2-coloring system, and which is non-normal iff the system is non-uniformizable.

Construction 4.1. Let $\mathcal{C}$ be a coherent 2-coloring system on ω_1. List $A = (\omega_1 \setminus \Lambda) \times \{0,1\}$ in lexicographical order, $A = \{a_\xi : \xi \in \omega_1\}$. For each $\alpha \in \Lambda$ let

$$B(\alpha, 0) = \{\langle \beta, 0\rangle : \beta \in \Lambda, \beta \leq \alpha\} \cup \{a_\xi : \xi \in W_\alpha\}.$$

This gives a sprat $X_\mathcal{C}$ in which $F_0 = \{\langle \alpha, 0\rangle : \alpha \in \Lambda\}$ and $F_1 = \{\langle \alpha, 1\rangle : \alpha \in \Lambda\}$ are disjoint copies of ω_1. If $\mathcal{C}$ is uniformizable by C_{ω_1}, then $F_0 \cup \{a_\xi : \xi \in W_{\omega_1}\}$ and $F_1 \cup \{a_\xi : \xi \in W_{\omega_1}\}$ are complementary clopen sets. Conversely, if V_0 and V_1 are complementary clopen sets containing F_0 and F_1 respectively, and $W = \{\xi : a_\xi \in V_0\}$, $B = \{\xi : a_\xi \in V_1\}$, then (W, B) is a uniformization of $\mathcal{C}$.

This construction of $X_\mathcal{C}$ has an inverse. Given any sprat Y in which $\Lambda \times \{0\}$ and $\Lambda \times \{1\}$ are disjoint copies of ω_1, and an assignment of 0th neighborhoods $B(\eta, i)$, we can define $\mathcal{C}_Y$ by letting $(W_Y)_\eta = \{\xi : a_\xi \in B(\eta, 0)\}$ for all $\eta \in \omega_1$. It is routine to verify that $X_{\mathcal{C}_Y} = Y$ and $\mathcal{C}_{X_\mathcal{D}} = \mathcal{D}$ for all coherent 2-coloring systems $\mathcal{D}$ and all sprats Y described in this paragraph.

5. Special classes of sprats

Definition 2. A sprat is *banded* if $B(\alpha, i)$ can be defined to contain every fiber $\pi^{\leftarrow}\{\xi\}$ that it meets whenever $\xi < \alpha$.

Definition 3. A sprat is *symmetrical* if $B(\alpha, i)$ can be defined to meet every fiber $\pi^{\leftarrow}\{\xi\}$ $(\xi \leq \alpha)$ in exactly one point.

Clearly Example 3.1 is symmetrical while Example 3.2 is banded. Example 3.3 is neither but can easily be shown homeomorphic to a banded sprat with the help of the Classical Theorem.

Lemma 5.1. *No banded sprat contains two disjoint copies of ω_1 but if a symmetrical sprat contains a copy of ω_1 it contains two disjoint copies.*

Corollary 5.2. *Every banded sprat is normal, while a symmetrical sprat with a copy of ω_1 is normal iff it is homeomorphic to $\omega_1 \times 2$.*

In Section 8 there is a ZFC example of a non-normal symmetrical sprat with two disjoint copies of ω_1. [Note that by Theorem 3.6, the last clause is redundant!] D. H. Fremlin [4] showed that the PFA implies every sprat contains a copy of ω_1. This has the consequence:

Theorem 5.3. *The PFA implies that for every sprat X there exists a club C such that $X \restriction C$ is either banded or symmetrical. In the former case, $X \restriction C$ is homeomorphic to (the Alexandroff duplicate of) ω_1.*

Problem 2. Does the PFA imply that every sprat is homeomorphic to either a banded sprat or a symmetrical sprat?

Under $\diamondsuit$ there is a wealth of sprat spaces that have no copies of ω_1 and are not homeomorphic to either a banded sprat or a symmetrical sprat, nor are their restrictions to any club.

In [4], Fremlin published an example (due to the author) of a banded sprat without a copy of ω_1 that exists in any model of ZFC obtained from a model of $\clubsuit$ by ccc forcing. The original Solovay-Tenenbaum model of MA(ω_1) is such a model.

Definition 4. A continuous preimage X of ω_1 is *monolithic* if every closed unbounded subset of X contains a preimage of a club.

Lemma 5.4. *Every monolithic perfect preimage of ω_1 is hereditarily collectionwise normal and hereditarily countably paracompact.*

Theorem 5.5. *A symmetrical sprat is monolithic iff it does not contain a copy of ω_1.*

There are easy examples under ♣ of monolithic symmetrical sprat spaces, and I recently showed they also exist under an axiom compatible with MA $+\neg$CH; see Section 8. In Section 7 we will show that monolithic banded sprats exist under the following axiom, recently shown by Hernandez and Ishiu to be compatible with MA(ω_1):

Axiom 1. There is a ladder system $\{L_\alpha : \alpha \in \Lambda \cap \omega_1\}$ such that for every club set C there is a club subset $K(C)$ such that $L_\alpha \subset^* C$ whenever $\alpha \in K(C)$. [$A \subset^* B$ means that $A \setminus B$ is finite.]

6. Martin's axiom and T_5 vs. hereditarily strongly cwH

Theorem 6.1. *If $2^{\aleph_0} < 2^{\aleph_1}$, then Example 3.3 is never T_5.*

Theorem 6.2. *If MA(ω_1), then Example 3.3 is always T_5.*

Proof of Theorem 6.1: Recall a theorem of Devlin and Shelah [2]: if $\mathcal{L}$ is a ladder system on a club subset of ω_1, then there is a piecewise monochromatic 2-coloring of $\mathcal{L}$ which cannot be uniformized if $2^{\aleph_0} < 2^{\aleph_1}$. $\square$

Part of the key to Theorem 6.2 is the theorem [2] that MA(ω_1) implies every 2-coloring of a ladder system on ω_1 is uniformizable. Another part is that the following axiom is a consequence of MA(ω_1):

Axiom 2. If $\mathcal{L} = \{L_\alpha : \alpha \in \omega_1 \cap \Lambda\}$ is a ladder system on ω_1 and C is a club subset of Λ, there is a choice of a cofinite $K_\alpha \subset L_\alpha$ for each $\alpha \in C'$ so that $\bigcup\{K_\alpha : \alpha \in C'\}$ meets each interval between successive members of C in a finite set.

Here C' denotes the derived set of C.

Theorem 6.3. *MA(ω_1) implies Axiom 2.*

Proof. Let $(\mathfrak{P}, \leq)$ be the following poset. Elements of $\mathfrak{P}$ are pairs $(\mathcal{A}, \mathcal{B})$ such that:

(1) $\mathcal{A}$ is a finite collection $K_{\alpha_1}, \dots K_{\alpha_n}$ where the α_i are distinct members of C' and each K_{α_i} is a cofinite subset of L_{α_i} and

(2) $\mathcal{B}$ is a finite collection of intervals (c_0, c_1) where c_0 and c_1 are successive elements of C.

The order on $\mathfrak{P}$ is given by:

(3) $(\mathcal{A}_0, \mathcal{B}_0) \leq (\mathcal{A}_1, \mathcal{B}_1) \iff \mathcal{A}_0 \subset \mathcal{A}_1$, $\mathcal{B}_0 \subset \mathcal{B}_1$, and $(\bigcup \mathcal{A}_1) \cap (\bigcup B_0) = (\bigcup \mathcal{A}_0) \cap (\bigcup B_0)$.

The proof that $\mathfrak{P}$ is $\uparrow$-c.c.c. uses classical techniques except perhaps in the last step. Given an uncountable subset of $\mathfrak{P}$, we can cut it down to an uncountable subset in which the first and second coordinates are all of size n_0 and n_1, respectively, and form Δ-systems with roots $\mathcal{A}_r$ and $\mathcal{B}_r$ respectively. We can further cut it down to an uncountable subset $\mathfrak{C}$ in which all the "leaves" $\mathcal{A} \setminus \mathcal{A}_r$ are ladders to ordinals greater than those to which the ladders in $\mathcal{A}_r$ converge, and similarly each $\mathcal{B} \setminus \mathcal{B}_r$ consists of intervals beginning further up than the intervals in $\mathcal{B}_r$.

Cut $\mathfrak{C}$ down to an uncountable $\mathfrak{D}$ so that for each $(c, c') \in \mathcal{B}_r$ there is a finite subset F_c such that if $(\mathcal{A}, \mathcal{B}) \in \mathfrak{D}$ then $[\bigcup A \cap (c, c')] = F_c$, and such that either (1) $\mathcal{B} = \mathcal{B}_r$ for all $(\mathcal{A}, \mathcal{B}) \in \mathfrak{D}$ or (2) $\mathcal{A} = \mathcal{A}_r$ for all $(\mathcal{A}, \mathcal{B}) \in \mathfrak{D}$ or (3) distinct members of $\mathfrak{D}$ have both distinct first coordinates and distinct second coordinates, and if $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}', \mathcal{B}')$ are distinct members of $\mathfrak{D}$ then the ordinals on which the members of $\mathcal{A} \setminus \mathcal{A}_r$ are ladders are all different from each of the ordinals on which the members of $\mathcal{A}' \setminus \mathcal{A}_r$ are ladders.

If (1) holds then any two members of $\mathfrak{D}$ are $\uparrow$-compatible. If (2) holds then all $(\mathcal{A}_r, \mathcal{B}) \in \mathfrak{D}$ such that $c > sup(\bigcup A_r)$ for all $(c, c') \in \mathcal{B} \setminus \mathcal{B}_r$ are pairwise compatible.

Finally, if (3) holds, cut $\mathfrak{D}$ down further to a family $\{(\mathcal{A}_\alpha, B_\alpha) : \alpha < \omega_1\}$ such that whenever $\alpha < \beta$ then every element of $\bigcup(\mathcal{A}_\alpha \cup \mathcal{B}_\alpha)$ is $<$ every element of $\bigcup(\mathcal{B}_\beta \setminus \mathcal{B}_r)$. Then $(A_\alpha, \mathcal{B}_\alpha)$ can only be $\uparrow$-incompatibile with $(A_\beta, \mathcal{B}_\beta)$ if $\bigcup(\mathcal{A}_\beta \setminus \mathcal{A}_r)$ meets $\bigcup(\mathcal{B}_\alpha \setminus \mathcal{B}_r)$. But $\bigcup \mathcal{A}_\beta$ is of order type ω and hence $\mathcal{A}_{\omega+1}$ must miss some $\bigcup(\mathcal{B}_\alpha \setminus \mathcal{B}_r)$ $(\alpha < \omega + 1)$. This completes the proof that $\mathfrak{P}$ has the $\uparrow$-c.c.c.

Using MA(ω_1), we obtain a subset $\mathfrak{G}$ of $\mathfrak{P}$ generic for the following dense sets:

$$\mathfrak{D}_B = \{(\mathcal{A}, \mathcal{B}) : B \in \mathcal{B}\} \quad (B = (c, c') \text{ for successive } c, c' \in C)$$

$$\mathfrak{D}_\alpha = \{(\mathcal{A}, \mathcal{B}) : K \in \mathcal{A}, K \text{is a cofinite subset of } L_\alpha\} \quad (\alpha \in \Lambda_2 \cap C')$$

Let $\mathfrak{G}$ be generic for these sets, and for each $\alpha \in \Lambda$ pick any $(\mathcal{A}(\alpha), \mathcal{B}(\alpha)) \in \mathfrak{G}$ such that some cofinite subset M_α of L_α is in $\mathcal{A}$. Since any two members of $\mathfrak{G}$ are $\uparrow$-compatible, M_α is uniquely determined. Next, given a pair (c, c') of successive members of C, pick any $(\mathcal{A}, \mathcal{B}) \in \mathfrak{G}$ such that $(c, c') \in \mathcal{B}$. Then any $(\mathcal{A}', \mathcal{B}')$ compatible with $(\mathcal{A}, \mathcal{B})$ must have $[\bigcup \mathcal{A}' \cap (c, c')] \subset [\bigcup \mathcal{A} \cap (c, c')]$, with equality if $(c, c') \in \mathcal{B}'$. And $[\bigcup \mathcal{A} \cap (c, c')]$ is finite because there is no member of C' in $(c, c']$ and $\mathcal{A}$ is a finite collection of ladders based on points of C'. $\square$

Proof of Theorem 6.2

7. Constructions of banded sprat spaces

Given a ladder system $\mathcal{L}$ on S, we define the banded sprat space $X_\mathcal{L}$ by induction on $\alpha \in \omega_1$. For finite α, we let $B(\alpha, 1) = \{\langle \alpha, 1 \rangle\}$ and, as standard, let $B(\alpha, 0)$ be the complement of $B(\alpha, 1)$ in $[0, \alpha] \times \{0, 1\}$. If α is not a limit point of S, we do the same. If α is a limit point of S, then we define:

$$B(\alpha, 1) = \{\langle \alpha, 1 \rangle\} \cup \bigcup_{n=0}^{\infty} (\sigma_\alpha(2n), \sigma_\alpha(2n+1)] \times \{0, 1\}.$$

Hence we also have:

$$B(\alpha, 0) = \{\langle \alpha, 0 \rangle\} \cup ([0, \sigma_0] \times \{0, 1\}) \cup \bigcup_{n=1}^{\infty} (\sigma_\alpha(2n-1), \sigma_\alpha(2n)] \times \{0, 1\}.$$

This construction is universal for banded sprats: it is possible, given a banded sprat space X, to find a ladder system on a subset of ω_1 such that $X_\mathcal{L}$ is homeomorphic to X. In fact, every sprat space is homeomorphic to one in which $\langle \alpha, 0 \rangle$ is never isolated for any limit ordinal α. In a banded sprat space like this, a ladder system for $X_\mathcal{L}$ arises naturally for any given choice of $\{B(\alpha, 0) : \alpha \in \omega_1 \cap \Lambda\}$.

To simplify the description, we put the fiber over 0 in $B(\alpha, 0)$ for all α; this obviously does not affect the topology. Each $B(\alpha, 0)$ consists, except for its last point $\langle\alpha, 0\rangle$, of complete fibers $\{\xi\} \times \{0, 1\}$. If all fibers over ordinals $< \alpha$ are represented, or more generally if there is some $\xi < \alpha$ such that all fibers strictly between ξ and α are in $B(\alpha, 0)$, then we leave α out of S altogether.

Otherwise, we let $\sigma_\alpha(0)$ be the supremum of the ordinals ξ such that all fibers over ordinals $\leq \xi$ are in $B(\alpha, 0)$. This supremum $\sigma_\alpha(0)$ is less than α because we are in the "Otherwise" case. Moreover, this supremum is actually a maximum, because $B(\alpha, 0)$ is closed and compact: at least one $\langle\sigma_\alpha(0), i\rangle$ has to be in the closure of $B(\alpha, 0)$ and since $\sigma_\alpha(0) < \alpha$ the whole fiber is in $B(\alpha, 0)$.

With $\sigma_\alpha(n) < \alpha$ defined, and the fiber over it in $B(\alpha, i)$, let $\sigma_\alpha(n+1)$ be the greatest $\xi > \sigma_\alpha(n)$ such that $\pi^{\leftarrow}\{\eta\} \in B(\alpha, 1-i)$ whenever $\sigma_\alpha(n) < \eta \leq \xi$. As with $\sigma_\alpha(0)$, a greatest such ξ exists and is less than α.

In this way, the fibers over successive $\sigma_\alpha(n)$ keep alternating between $B(\alpha, 0)$ and $B(\alpha, 1)$. Both $B(\alpha, 0)$ and $B(\alpha, 1)$ are compact, so the fiber over $\sup\{\sigma_\alpha(n) : n \in \omega\}$ contains points of both sets. But this is possible only if $\sup\{\sigma_\alpha(n) : n \in \omega\}$ is α. Thus σ_α is a ladder at α, and if we define it for all limit α, the resulting ladder system $\mathcal{L}$ on ω_1 gives us back our space X with $X_{\mathcal{L}}$.

Example 7.1. If $\mathcal{L}$ witnesses Axiom 1, then the resulting banded sprat is monolithic.

8. Constructions of symmetrical sprat spaces

For symmetrical sprat spaces there is no ladder system as intimately connected with the topology as the one for banded sprats. As partial compensation, we have the following property for symmetrical sprats, due to the fact that $B(\alpha, 0)$ and $B(\alpha, 1)$ split every fiber over $[0, \alpha]$ between them: if $\beta < \alpha$ and $\langle\beta, i\rangle$ is in $B(\alpha, 0)$ then there exists $\gamma < \beta$ such that

$$B(\alpha, 0) \cap \pi^{\leftarrow}(\gamma, \beta] = B(\beta, i) \cap \pi^{\leftarrow}(\gamma, \beta].$$

Of course this is equivalent to $B(\alpha, 1) \cap \pi^{\leftarrow}(\gamma, \beta] = B(\beta, 1-i) \cap \pi^{\leftarrow}(\gamma, \beta]$. We express this by saying that "*α respects β on $(\gamma, \beta]$*."

It is easy to construct a ladder σ_α at any limit ordinal with the property that α respects $\sigma_\alpha(0)$ on $[0, \sigma_\alpha(0)]$ and also respects $\sigma_\alpha(n+1)$ on $(\sigma_\alpha(n), \sigma_\alpha(n+1)]$. One can also use ladders in the opposite way, as in the following example.

Example 8.1. This is the promised ZFC example of a non-normal symmetrical sprat space. The key idea is (1) to define $B(\alpha, 0)$ so that it contains all $\langle\beta, 0\rangle$ whenever β is a limit ordinal $\leq \alpha$ and (2) to define $B(\alpha, 0)$ at limits of limit ordinals in such a way that if $\langle\beta_n\rangle_{n\in\omega}$ is a sequence of limit ordinals converging to α, and α respects β_n on $(\xi_n, \beta_n]$ then the sequence of ξ_n converges to α.

To see that this strategy works, we first note that (1) implies that the relative topology on $\pi^{\leftarrow}\Lambda$ is the product topology and so $\Lambda \times \{0\}$ and $\Lambda \times \{1\}$ are disjoint closed copies of ω_1. Because of this, the inverse of Construction 4.1 gives us a coherent 2-coloring system on ω_1 which is not uniformizable.

Next, (2) implies that if U is an open set containing $\Lambda \times \{0\}$, and $\gamma(\alpha) < \alpha$ is chosen for each limit ordinal so that $B(\alpha, 0) \cap \pi^{\leftarrow}(\gamma(\alpha), \alpha] \subset U$, the pressing-down lemma gives γ such that $\gamma = \gamma(\alpha)$ for uncountably many α. However, due to insufficient respect between the various α's, U must contain the fiber over uncountably many ordinals, and so the closure of U meets $\Lambda \times \{1\}$, contradicting normality.

To implement this strategy, we let $\langle \sigma_\alpha : \alpha \in \Lambda \rangle$ be a ladder system such that $\sigma_\alpha(0) = 0$ for all $\alpha \in \Lambda$ and we define each $B(\alpha, 0)$ by induction to satisfy the stronger condition:

(2^+) If $\beta \in \Lambda,\ \alpha \in \Lambda_2, \sigma_\alpha(k) < \beta \leq \sigma_\alpha(k+1)$,
then α does not respect β on $(\sigma_\alpha(k), \beta)$.

In a forthcoming paper, we will show that the following axiom is enough to produce a monolithic symmetrical sprat space.

Axiom 3. There is a base $\mathcal{B}$ for the club filter on ω_1 such that $\mathcal{B} \restriction \alpha (= \{B \cap \alpha : B \in \mathcal{B}\})$ is countable for all $\alpha \in \omega_1$.

This axiom has been tentatively designated KH^+ because it obviously implies the existence of Kurepa families (and hence of Kurepa trees) as defined in [6], and because it is easily shown to be satisfied by any ccc forcing extension of a model of $\diamondsuit^+$. Hence it is compatible with $MA(\omega_1)$.

The construction of the monolithic example is a refinement of the technique in Example 8.1. At each limit stage α, we list the countably many sets in $B \restriction \alpha$ that are unbounded in α, and in the nth step in the construction at stage α we make sure that the first n members of $B \restriction \alpha$ get both points over α in the closure. The reason we do not do the construction here in detail is that it is a prototype for a more sophisticated construction of a hereditarily collectionwise normal, countably compact 2-manifold compatible with $MA(\omega_1)$, and the two are best presented together. The manifold construction is in contrast to the main theorem of [7], which is that the PFA (which implies $MA(\omega_1)$) implies that every normal, hereditarily strongly cwH manifold of dimension > 1 is metrizable.

9. Partial uniformization of coherent 2-coloring systems on ω_1.

Coherent 2-coloring systems closely correspond to P-ideals on ω_1 whose ω-orthocomplements are also P-ideals:

Definition 9.1. A collection $\mathcal{I}$ of countable subsets of a set X is a *P-ideal* if it is downward closed with respect to $\subset$, closed under finite union, and has the property that, if $\{I_n : n \in \omega\}$ is a countable subset of $\mathcal{I}$, then there exists $J \in \mathcal{I}$ such that $I_n \subset^* J$ for all n. [Here, $A \subset^* B$ means $A \setminus B$ is finite.]

Definition 9.2. Given an ideal $\mathcal{I}$ of subsets of a set S, a subset A of S is *orthogonal to* $\mathcal{I}$ if $A \cap I$ is finite for each $I \in \mathcal{I}$. The *ω-orthocomplement of* $\mathcal{I}$ is the ideal $\{J : |J| \leq \omega,\ J \text{ is orthogonal to } \mathcal{I}\}$ and will be denoted $\mathcal{I}^\perp$.

When restricted to ideals whose members are countable, ω-orthocomplementation is a Galois correspondence, which means that it is order-reversing and each ideal is a subideal of its double dual. That is, if $\mathcal{I} \subset \mathcal{J}$ then $\mathcal{J}^\perp \subset \mathcal{I}^\perp$ and we have

$\mathcal{I} \subset \mathcal{I}^{\perp\perp}$. As in all Galois correspondences, this has the easy consequence that $\mathcal{I}^{\perp} = \mathcal{I}^{\perp\perp\perp}$.

If $\mathcal{A} = \{A_\alpha : \alpha \in \omega_1\}$ is a coherent sequence on ω_1, it generates a P-ideal on ω_1, and its ω-orthocomplement on ω_1 is generated by $\mathcal{A}^* = \{\alpha \setminus A_\alpha : \alpha \in \omega_1\}$ together the singleton subsets of ω_1. This ω-orthocomplement is also a P-ideal.

To show the correspondence mentioned in the opening sentence of this section, we use the fact that $\mathcal{A}^*$ is itself a coherent sequence, and that both ideals are countable-covering:

Definition 9.3. An ideal $\mathcal{J}$ of subsets of a set X is *countable-covering* if for each $Q \in [X]^\omega$, the ideal $\mathcal{J} \restriction Q$ is countably generated.

In other words, for each countable subset Q of X, there is a countable subcollection $\{J_n^Q : n \in \omega\}$ of $\mathcal{J}$ such that every member J of $\mathcal{J}$ that is a subset of Q satisfies $J \subset J_n^Q$ for some n.

The P-ideals in all but the last section of [1], as well as the P-ideals in this paper, are all ω-orthocomplements of countable-covering ideals. There are no exceptions in the opposite direction:

Theorem 9.4. *The ω-orthocomplement of a countable-covering ideal is a P-ideal.*

Proof. Let $\mathcal{J}$ be a countable-covering ideal on the set S and let $\mathcal{I} = \mathcal{J}^{\perp}$. If $\{I_n : n \in \omega\} \subset \mathcal{I}$, let $Q = \cup\{I_n : n \in \omega\}$, and let $\{J_n : n \in \omega\}$ be as in Definition 9.3. Then by the Dubois-Reymond property of $\mathcal{P}(\omega)/\text{fin}$, there is a subset H of Q such that $I_n \subset^* H$ for all $n \in \omega$ while $J_n \cap H$ is finite for all n. It is easy to see that $H \in \mathcal{J}^{\perp}$, as required. $\square$

It follows that if $\mathcal{J}$ is a countable-covering ideal on ω_1 that is also a P-ideal, then its ω-orthocomplement also enjoys both properties. Moreover, for each α there is a single member J^α of $\mathcal{J}$ such that every member of $\mathcal{J} \restriction \alpha$ is almost contained in J^α (that is, $J \subseteq \mathcal{J}$ for each $J \in \mathcal{J} \restriction \alpha$). Thus $\mathcal{J}$ is generated by the coherent sequence $\{J^\alpha : \alpha \in \omega_1\}$ together with the singleton subsets of ω_1. Now to produce a coherent 2-coloring, we have C_α color the elements of J^α white and all the other elements of α black.

To get a better handle on the uniformizability of coherent 2-colorings, we recall the following axioms from [3]:

Definition 9.5. The following axiom is denoted *Axiom* P_{11}:
For every P-ideal $\mathcal{I}$ on a stationary subset S of ω_1, either

(i) there is an uncountable $A \subset S$ such that every countable subset of A is in $\mathcal{I}$, or

(ii) there is an uncountable $B \subset S$ such that every countable subset of B is in $\mathcal{I}^{\perp}$.

Axiom P_{12} [resp. *Axiom* P_{21}] substitutes "stationary" for "uncountable" in (ii) [*resp.* (i)], while *Axiom* P_{22} makes the same substitution in both (i) and (ii).

The following theorem is immediate from the definitions:

Theorem 9.6. *If P_{11}, then every coherent 2-coloring on ω_1 is uniformizable on some uncountable subset of ω_1.*

Clearly, we could have put CC_{11} in place of P_{11}:

Definition 9.7. *Axiom* CC_{11} is the axiom that for each countable-covering ideal $\mathcal{J}$ on a stationary subset S of ω_1, either:

(i) there is an uncountable $A \subset S$ such that $[A]^\omega \subset \mathcal{J}$; or

(ii) there is an uncountable $B \subset S$ such that $[B]^\omega \subset \mathcal{J}^\perp$.

Similarly, Axioms CC_{12}, CC_{21} and CC_{22} are defined analogously to the corresponding P_{mn} axioms. It is not hard to show that P_{ij} implies C_{ji} for $\{i, j\} \subset \{1, 2\}$. We can add 3 to the latter set once we define:

Definition 9.8. *Axiom* P_{31} [resp. *Axiom* CC_{13}] is the following axiom.

For every P-ideal $\mathcal{I}$ [*resp.* for every countably covering ideal $\mathcal{J}$] on ω_1, either

(i) there is a closed unbounded $A \subset \omega_1$ such that every countable subset of A is in $\mathcal{I}$ [resp. in $\mathcal{J}^\perp$], or

(ii) there is an uncountable $B \subset \omega_1$ such that every countable subset of B is in $\mathcal{I}^\perp$ [resp. in $\mathcal{J}$].

Axiom P_{32} [resp. *Axiom* CC_{23}] is obtained from Axiom P_{31} [resp. *Axiom* CC_{13}] by substituting "stationary" for "uncountable" in (ii).

We do not yet know whether any or all of the axioms in 9.8 are compatible with CH. This was claimed in an early version of [5], where it is designated "$(*_c)$ for $\theta = \omega_1$," but the proof was faulty. This proof was, however, adequate to show that these axioms follow from the PFA [5]. Also, substituting a stationary set for ω_1 in each case gives an axiom which is compatible with CH.

The following strengthening of P_{21} was the main axiom in [1]:

$(*)$ For every P-ideal $\mathcal{I}$ on ω_1, either

(i) there is an uncountable subset A of ω_1 such that every countable subset of A is in $\mathcal{I}$, or

(ii) ω_1 is the union of countably many sets $\{B_n : n \in \omega\}$ such that $B_n \cap I$ is finite for all n and for all $I \in \mathcal{I}$.

Of course, any set X of cardinality ω_1 can be substituted for ω_1 here, while if X is countable, (ii) is trivially satisfied. The axiom $(*)$ follows from the PFA and is compatible with CH.

Now suppose CC_{23} holds. If $\mathcal{W}$ is any nontrivial coherent sequence on ω_1, then we can apply CC_{23} to both $\mathcal{W}$ and $\mathcal{W}^*$, and get one of the following to hold:

Case I. There is a club subset C of ω_1 on which the coherent 2-coloring associated with $\mathcal{W}$ is uniformizable, with all of C colored either white or black.

Case II. There is a stationary set S on which the coherent 2-coloring associated with $\mathcal{A}$ is uniformizable, with the white points of S forming a stationary, co-stationary subset of S.

Either case can be broken down further. Suppose we are in Case I with the club C almost all white. Applying $(*)$ to the still nontrivial sequences $\mathcal{W} \restriction (\omega_1 \setminus C)$ and $\mathcal{W}^* \restriction (\omega_1 \setminus C)$, we either have:

Case IA: $\omega_1 \setminus C$ can be partitioned into countably many sets, all of which can be colored white, or all of which can be colored black, and cohere with the 2-coloring associated with $\mathcal{W}$.

Case IB: There is an uncountable subset Z of $\omega_1 \setminus C$ with a 2-coloring cohering with that associated with $\mathcal{W}$, in which uncountably many points of Z are white and uncountably many are black.

Case IA is definitive, but IB still leaves the restriction of $\mathcal{W}$ to $\omega_1 \setminus (C \cup Z)$ nontrivial. To streamline the subsequent analysis, it is helpful to note that $\mathcal{H} = \{H \in \omega_1 : \mathcal{W} \restriction H \text{ is trivial } \}$ is an ideal to which every set that is almost included in some member of $\mathcal{W}$ or of $\mathcal{W}^*$ belongs.

Definition 9.9. A coherent sequence $\mathcal{A}$ is *tame* [resp. *almost tame*] if ω_1 [*resp.* some club subset C of ω_1] is the union of countably many sets Z_n such that $\mathcal{A} \restriction Z_n$ is trivial.

Problem 3. Is it consistent that every coherent sequence is tame, or at least almost tame? Does either conclusion follow from $(*)$ or from a combination of $(*)$ with CC_{23}?

Problem 4. Does ZFC imply the existence of a coherent sequence $\mathcal{A}$ such that $\mathcal{A} \restriction C$ is nontrivial for each club $C \subset \omega_1$?

If the answer is negative, then a model that witnesses this would satisfy the restriction of CC_{23} to countable-covering ideals that are also P-ideals.

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Department of Mathematics, University of South Carolina, Columbia, SC,
E-mail address: `nyikos@math.sc.edu`