A collectionwise normal, hereditarily weakly θ-refinable Dowker space

This paper answers a question the author posed over three decades ago in an article on "classic problems" in the first issue of *Topology Proceedings* [1].

Question 1. *Is every collectionwise normal, weakly θ-refinable space paracompact?*

The best result heretofore on the question was a space of Peter de Caux obtained under an extra set-theoretic axiom:

Theorem 1. [2] *The axiom $\clubsuit$ implies the existence of a hereditarily weakly θ-refinable, collectionwise normal space that is not countably metacompact (hence not paracompact).*

This appeared in the same first issue of *Topology Proceedings*, and its presence there was noted in the "Classic Problems" paper [1], and so it was understood that Question 1 was asking either for an independence result or an example that did not use anything beyond the usual (ZFC) axioms. The following related question was posed there:

Problem 1. *Is every collectionwise normal, screenable space paracompact?*

At the time, no consistency results were known at all on Problem 1. Subsequently, Mary Ellen Rudin produced a counterexample [3] under the axiom $\diamondsuit^{++}$, but we still do not have a ZFC answer for it.

The main result of this paper is a ZFC answer to Question 1 that also satisfies a number of properties that de Caux's space did not satisfy:

Theorem 2. *There is a collectionwise normal, hereditarily weakly θ-refinable, hereditarily meta-Lindelöf, hereditarily realcompact space that is not countably metacompact (hence not paracompact).*

The $\clubsuit$ space of de Caux was not meta-Lindelöf and was not realcompact. In fact, it is still essentially the only known example of a weakly θ-refinable normal space that is known not to be realcompact, except for the well-known case of discrete spaces of measurable cardinality (and, of course, spaces with closed discrete subspaces of measurable cardinality). And so, the following related problem from [1] is still not completely solved:

Problem 2. *Is every normal θ-refinable space of non-measurable cardinality realcompact?*

[In this paper, "normal" is understood to include the T_1 property, and so it implies the space is Tychonoff.]

Back in 1978, general topologists were still using a definition of "measurable cardinality" that included all cardinals greater than the first measurable cardinal. No consistency results are known for the following problem:

Problem 3. *Is every normal screenable space of non-measurable cardinality realcompact?*

It is not known whether the screenable Dowker spaces in [4] and [5] are realcompact, nor whether the $\diamondsuit^{++}$ example [3] is realcompact.

The space used to prove Theorem 2 is a variation on a space by Balogh [6], who used it to show:

Theorem 3. *There is a hereditarily collectionwise normal, hereditarily meta-Lindelöf, hereditarily realcompact space that is not countably metacompact (hence not paracompact).*

The hereditary realcompactness was built into Balogh's space at the outset, and it was done in order to answer a question posed in Mary Ellen Rudin's 1971 paper [7] where she first constructed a Dowker space in ZFC:

Question 2. *Is there a realcompact Dowker space?*

[A **Dowker space** is a normal space whose product with $[0, 1]$ is not normal. Equivalently, it is a normal space which is not countably metacompact. Other topological concepts used above are defined at the end of this section.]

Realcompactness was built into the main example of this paper because it was easy to do so, and because the author was unable to tell whether the space resulting from omitting the extra open sets used to establish realcompactness is realcompact or not. This is also true of Balogh's space that was used as a prototype.

In modifying Balogh's space, it seemed necessary to sacrifice hereditary collectionwise normality in order to answer Question 1. In many ways, it would have been preferable to do it the other way around — remove "hereditarily" from in front of "θ-refinable" while keeping it in front of "collectionwise normal." This is due to the fact that the resulting space would also have answered Problem 1, because:

Theorem 4. *Every hereditarily collectionwise normal, weakly θ-refinable space is screenable.*

However, there are major reasons why this did not happen, some of which will be pointed out in the course of proving that the construction here satisfies the demanding conditions of Theorem 2. We close this section with some standard definitions.

Definition 1. A *weakly θ-refinable* space is a space X in which every open cover has an open refinement $\mathcal{W} = \bigcup_{n=0}^{\infty} \mathcal{W}_n$ such that for each point $x \in X$ there exists n such that $1 \leq ord(x, \mathcal{W}_n < \aleph_0$.

Definition 2. A *paracompact meta-Lindelöf*] space is one in which every open cover has a locally finite [*resp.* point-countable] open refinement.

Definition 3. *A screenable* [resp. *σ-metacompact*] space is one in which every open cover has a σ-disjoint [*resp. σ-point-finite*] open refinement.

Here, as usual, the prefix σ- stands for being the union of countably many collections satisfying the property that follows. Paracompactness is equivalent to each open cover having a σ-discrete open refinement, giving the first implication below, and the other one is obvious:

paracompact $\implies$ screenable $\implies$ σ-metacompact

It is also obvious that these concepts imply both "meta-Lindelöf" and "weakly θ-refinable." There are well-known counterexamples to any implication between these concepts and the earlier ones that does not logically follow from these obvious facts.

Definition 4. A *collectionwise normal* space X is one in which, for each discrete collection $\{D_\gamma : \gamma \in \Gamma\}$ of subsets of X, there is a disjoint collection of open sets $\{U_\gamma : \gamma \in \Gamma\}$ such that $D_\gamma \subset U_\delta$ iff $D_\gamma = D_\delta$. A *countably paracompact* [resp. *countably metacompact* space is one in which each countable open cover has a locally finite [*resp.* point-finite] open refinement.

Definition 5. A space X is *realcompact* if it is Tychonoff and every ultrafilter in the lattice of zero-sets that has the countable intersection property (c.i.p.) is fixed, *i.e.,* has nonempty intersection.

The word "realcompact" comes from other characterizations of these spaces; the easiest is that they are the spaces that can be embedded as closed subspaces of some $\mathbb{R}^\kappa$.

1. The definition of the main example.

The exposition of the main example follows that of [6] very closely, though not as closely as [5] follows that in [4]. In particular, the proofs of several results are the same. They are repeated here with slight rewordings, it being a deplorable tendency of university libraries to cart old issues of journals off to relatively inaccessible annexes after a certain period of time.

There are, of course, significant differences between this paper and [6], and these will be pointed out from time to time, along with some similarities. However, there is no unified treatment treatment of the main example here and Balogh's main example, like there was in [5] of Balogh's screenable Dowker space [4] and the author's modifications thereof.

The underlying set for the main example X of this paper is $\mathfrak{c} \times \omega$. As in [6], we identify $\mathfrak{c}$ with the Cantor set and let $\{q_n : n \in \omega\}$ be a base for the usual topology on the Cantor set. We adopt the following notation: $L_n = \mathfrak{c} \times \{n\}$ and

$W_n = \mathfrak{c} \times n$ $(= \bigcup_{i=0}^{n-1} L_i)$. In particular, $W_0 = \emptyset$. In the base $\mathfrak{B}$ for the topology τ_w, we include $X \setminus \{x\}$ in $\mathfrak{B}$ for each $x \in X$ and $D_x = \{x\} \cup W_n$ for each $x \in L_n$. The sets $X \setminus \{x\}$ ensure that X is a T_1 space, while the D_x ensure that each W_n open in X and that each L_n discrete in its relative topology. The base $\mathfrak{B}$ is defined by induction, beginning with

$$\mathfrak{B}_0 = \{D_x : x \in X\} \cup \{X \setminus \{x\} : x \in X\} \cup \{\pi^{\leftarrow}(q_n) : n \in \omega\}.$$

These sets are already enough to ensure that L_n is discrete in its relative topology inherited from any topology finer than the one for which $\mathfrak{B}_0$ forms a subbase. This in turn is enough to show:

Proposition 1.0.0. *$\langle X, \tau_w \rangle$ is hereditarily weakly θ-refinable.*

In [6], the base begins with

$$\mathfrak{B}_0 = \{B_x : x \in X\} \cup \{W_n : n \in \omega\} \cup \{\pi^{\leftarrow}(q_n) : n \in \omega\}.$$

[In both $\langle X, \tau_B \rangle$ and $\langle X, \tau_w \rangle$, $\mathfrak{B}_0$ and many other $\mathfrak{B}_\alpha$ are just subbases for a topology, but the final collection $\mathfrak{B}_{2^{\mathfrak{c}}} = \mathfrak{B}$ is a base.]

There is no need to include the sets W_n in the description of the $\mathfrak{B}_0$ that goes with $\langle X, \tau_w \rangle$, since they are already open in the topology for which this is a subbase.

Proposition 1.0.1. *Every point in X is a G_δ.*

Definition 1.0.2. Let $\langle \mathcal{S}_\alpha \rangle_{\alpha < 2^{\mathfrak{c}}}$ be a listing of two kinds of $\mathfrak{c}$-sequences of subsets of X, with each sequence that is indexed by an infinite ordinal listed $2^{\mathfrak{c}}$ times. Both kinds are sequences of subsets of X, with the ones indexed by $n < \omega$ particularly simple:

$$\mathcal{S}_n = \langle \{ \langle W_n \cup \{ \langle \rho, n+1 \rangle \} : \rho \in \mathfrak{c} \rangle.$$

Clearly, $\mathcal{S}_n$ is one-to-one, and its range is $\{D_x : x \in L_n\}$.

The $\mathcal{S}_\xi$ for infinite ξ are essentially Type I sequences in the terminology of [6]. They are sequences of disjoint subsets,

$$\mathcal{S}_\xi = \langle F_\xi^\rho : \rho < \mathfrak{c} \rangle$$

where $F_\xi^\rho = \emptyset$ is allowed, even for all but finitely many ρ.

For finite n, the sequences $\mathcal{S}_n$ correspond to the Type II sequences of [6]. They will play the same role here as the Type II sequences did there, to ensure that X is hereditarily meta-Lindel´of. Although they come first in our listing for τ_w, they will be referred to as Type II sequences to stay close to what one finds in [6].

The base $\mathfrak{B}$ for (X, τ_w) is built in a $2^{\mathfrak{c}}$-step induction, with an ascending sequence of subbases $\mathfrak{B}_\alpha$. We use the term α**-open** to denote "open in the topology generated by $\mathfrak{B}_\alpha$." Each $\mathcal{S}_n$ $(n \in \omega)$ will be refined to a point-countable collection of subsets, $\langle V_n^\rho : \rho < \mathfrak{c}\rangle$, of W_{n+1}, giving $\mathfrak{B}_{n+1} = \mathfrak{B}_n \cup \{V_n^\rho : \rho < \mathfrak{c}\}$. This builds in the hereditary meta-Lindelöf property for τ_w [1.1.1 below].

In [6], the Type I sequences were of the form

$$\mathcal{S}_\xi = \langle O_\xi, \langle F_\xi^\rho \rangle_{\rho < \mathfrak{c}} \rangle$$

where O_ξ is a subset of X and the F_ξ^ρ are disjoint subsets of O_ξ. In that notation, our $\mathcal{S}_\xi$ would be $\langle X, \langle F_\xi^\rho \rangle_{\rho < \mathfrak{c}} \rangle$.

When $\xi \geq \omega$, $\mathcal{S}_\xi$ will be ignored until the first (if any) η such that $\mathcal{S}_\xi = \mathcal{S}_\eta$ and $\mathcal{S}_\eta$ is a discrete collection of η-closed subsets of X in the topology generated by $\mathfrak{B}_\eta$. Then it will be expanded to a disjoint collection of open sets B_η^ρ created for that purpose, with the goal of ensuring collectionwise normality [2.1 below]. Then, $\mathfrak{B}_{\eta+1}$ will be the union of $\mathfrak{B}_\eta$ with this disjoint expansion.

To simplify the construction, this will be done even if it is already possible to put the F_ξ^ρ into η-open sets.

In [6], Balogh's treatment of Type I sequences entailed waiting for a stage η (if any) where O_ξ was open in the topology generated by $\mathfrak{B}_\eta$, and the F_ξ^ρ constituted a discrete collection of closed sets in the relative topology of O_ξ wrt $\mathfrak{B}_\eta$. Then $\langle F_\xi^\rho \rangle_{\rho < \mathfrak{c}}$ was expanded to a disjoint collection of subsets B_η^ρ which were added to $\mathfrak{B}_\eta$ to produce $\mathfrak{B}_{\eta+1}$. This was done to ensure hereditary collectionwise normality of (X, τ_B), using the well-known fact that a space is hereditarily [collectionwise] normal iff every open subspace is [collectionwise] normal. For τ_w, it seemed necessary to settle for just collectionwise normality. The reasons for this will be given in a series of numbered remarks.

In [6], Balogh had $2^{\mathfrak{c}}$ Type II sequences; these were sequences $\langle U_\xi^\rho \rangle$ of subsets of X indexed by $\mathfrak{c}$, $(\xi < 2^{\mathfrak{c}})$ and a similar waiting game was used for them as for Type I sequences: one waited for a stage when all the members of the sequence were open; when and if this occurred, the sequence was refined to a point-countable collection of sets covering $\bigcup_{\rho < \mathfrak{c}} V_\xi^\rho$ in a way that generalizes our refinement of $\mathcal{S}_n$ $(n \in \omega)$. [See below.]

For each $A \in [X]^\omega$ and every $\alpha \in \mathfrak{c}$, let $\mathcal{S}_\alpha \restriction A$ be the sequence $\langle S_\alpha^\rho \cap A : \rho < \mathfrak{c}\rangle$. Let $\mathfrak{S}(A)$ be the collection of all sequences $\mathcal{S}_\xi \restriction A$ as ξ ranges over the infinite ordinals $< \mathfrak{c}$.

Definition 1.1. A triple $\langle A, \mathcal{D}, u\rangle$ is called a **control triple** if it satisfies:

(C_1)) A is a countably infinite subset of X;

(C_2) $\mathcal{D}$ is a countable subcollection of $\mathfrak{S}(A)$.

(C_3) u is a function with $\mathrm{dom}(u) \in [A]^\omega$ such that, for all $x \in \mathrm{dom}(u)$, $u(x)$ is a countable subcollection of $\mathfrak{S}(A)$ that is disjoint from $\mathcal{D}$;

(C_4) $x \neq x'$ in $\mathrm{dom}(u)$ implies $u(x) \cap u(x') = \emptyset$.

Let $\langle A_\beta, \mathcal{D}_\beta, u_\beta\rangle_{\beta<\mathfrak{c}}$ be a listing of all control triples, listing each one $\mathfrak{c}$ times in every open subset of the Cantor set (equivalently, in each $q(n)$ described above). The first ω steps are simple. We refine $\mathcal{S}_n$ by replacing each member $W_n \cup \{\langle\rho, n\rangle\}$ of the range of $\mathcal{S}_n$ with a set V_n^ρ such that $\langle\rho, n\rangle \in V_n^\rho \subset W_n \cup \{\langle\rho, n\rangle\}$. For the points $\langle\beta, i\rangle$ of W_n, we let $\langle\beta, i\rangle \in V_n^\rho$ if, and only if, $\rho \in \pi(A_\beta)$. Then for each finite n we have

$$\mathfrak{B}_{n+1} = \mathfrak{B}_0 \cup \bigcup_{k=1}^{n} \{V_k^\rho : \rho < \mathfrak{c}\}$$

and we have $\mathfrak{B}_\omega = \mathfrak{B}_0 \cup \bigcup_{k=1}^{\infty} \{V_k^\rho : \rho < \mathfrak{c}\}$.

Proposition 1.1.1. *(X, τ_w) is hereditarily meta-Lindelöf. That is, if Z is a subspace of (X, τ_w) then Z is meta-Lindelöf.*

Proof. The following argument works for any (X, τ) finer than the topology generated by $\mathfrak{B}_\omega$.

Let $\mathcal{U}$ be an open cover of Z, and for each $z = \langle\beta, n\rangle \in Z$, pick $U \in \mathcal{U}$ such that $z \in U$ and let $V_z = V_n^\beta \cap U$. Then $\{V_z : z \in L_n\}$ is a point-countable collection of (relatively) open subsets of Z, and so $\{V_z : z \in Z\}$ is a point-countable refinement of $\mathcal{U}$. $\square$

Remark 1. In any hereditarily collectionwise normal (X, τ) finer than the topology generated by $\mathfrak{B}_0$, the "horizontal line" L_n is closed discrete in the relative topology of the 0-open set W_{n+1}, and so it would expand to a disjoint collection of τ-open subsets of X. The proof of 1.1.1 would then give hereditary screenability of $\langle X, \tau\rangle$ all by itself. However, the arguments used here and in [6] for showing that X is not countably metacompact run into formidable obstacles in any such modification, as will be pointed out at the appropriate times. Efforts to find alternative methods of proving non-countable metacompactness, or of constructing a collectionwise normal, screenable Dowker space in ZFC have thus far been unsuccessful.

The expansions of the $\mathcal{S}_\xi$ for infinite ξ to disjoint open collections is done by induction on ξ and also on the columns C_β ($\beta \in \mathfrak{c}$) for a given ξ, where $C_\beta = \{\beta\} \times \omega$. In the process, we define $\mathfrak{B}_\xi$ for $\xi \geq \omega$ by letting $\mathfrak{B}_\xi = \bigcup_{\eta<\xi} \mathfrak{B}_\eta$ if ξ is a limit ordinal and $\mathfrak{B}_{\eta+1} = \mathfrak{B}_\eta \cup \{B_\xi^\rho : \rho < \mathfrak{c}\}$ for all η.

So, suppose we have defined $\mathfrak{B}_\xi$. The definition of $\mathfrak{B}$ and hence of τ_w will be clear once we define $\langle B_\xi^\rho : \rho < \mathfrak{c}\rangle$.

Case 1. $\mathcal{S}_\xi = \langle F_\xi^\rho \rangle_{\rho<\mathfrak{c}}$ is a discrete sequence of closed subsets of X in the topology generated by $\mathfrak{B}_\xi$, and ξ is minimal for this to be true. [For instance, it could be that there is $\eta < \xi$ such that $\mathcal{S}_\eta = \mathcal{S}_\xi$ and each F_ξ^ρ is closed in the topology generated by $\mathfrak{B}_\eta$, but the collection is not yet discrete in this coarser topology.]

Suppose $\beta < \mathfrak{c}$ and it has been determined which B_ξ^ρ the points of $\bigcup_{\alpha<\beta} C_\alpha$ are to be distributed among.

Subcase 1.1. If $\mathcal{S}_\xi \restriction A_\beta \in \mathcal{D}_\beta$,, we distribute the points of C_β as follows.

(a) If $F_\xi \cap C_\beta = \emptyset$, we make $B_\xi \cap C_\beta = \emptyset$. In other words, we leave all $\langle \beta, k\rangle$ out of all the B_ξ^ρ.

(b) If $F_\xi \cap C_\beta \neq \emptyset$, and $\langle \beta, i\rangle \in F_\xi^\rho$ for some ρ then we (have to) put $\langle \beta, i\rangle$ into B_ξ^ρ, and we define $\langle \beta, i\rangle(\xi) = \rho = \overline{\langle \beta, i\rangle}(\xi)$. As for the other points of C_β, we let j be the least integer for which $\langle \beta, j\rangle \in F_\xi$, let σ be defined by $\langle \beta, j\rangle \in F_\xi^\sigma$, put the points not in F_ξ into B_ξ^σ, and let $\langle \beta, i\rangle(\xi) = \sigma = \overline{\langle \beta, j\rangle}(\xi)$ for these points.

[More formally, (b) says that:

(i) if $\langle \beta, i\rangle \in F_\xi \setminus F_\xi^{\overline{\langle \beta, j\rangle}(\xi)}$, then $\langle \beta, i\rangle \in B_\xi^{\overline{\langle \beta, i\rangle}(\xi)}$ and

(ii) if $\langle \beta, i\rangle \in (X \setminus F_\xi) \cup F_\xi^{\overline{\langle \beta, j\rangle}(\xi)}$, then $\langle \beta, i\rangle \in B_\xi^{\overline{\langle \beta, j\rangle}(\xi)}$]

Subcase 1.2. There is an $x = \langle \alpha, n\rangle \in \mathrm{dom}(u_\beta) \subset A$ such that $\alpha < \beta$, $\mathcal{S}_\xi \restriction A_\beta \in u_\beta(x)$, and x has been put into B_ξ^γ at the αth step in this induction. [Note that $\mathcal{S}_\xi \restriction A \notin \mathcal{D}_\beta$ by (C_3) and that there can be only one such x by (C_4).]

(a) If $\langle \beta, i\rangle \notin F_\xi$, then we put $\langle \beta, i\rangle$ into B_ξ^γ and thus let $\langle \beta, i\rangle(\xi) = \gamma$.

(b) If $\langle \beta, i\rangle \in F_\xi^\rho$ for some ρ then we (have to) put $\langle \beta, i\rangle$ into B_ξ^ρ, making $\langle \beta, i\rangle(\xi) = \rho = \overline{\langle \beta, i\rangle}(\xi)$ for these points.

Subcase 1.3. If neither Subcase 1.1 nor Subcase 1.2 holds, then if $\langle \beta, i\rangle \notin F_\xi$, we put $\langle \beta, i\rangle$ into B_ξ^0; while if $\langle \beta, i\rangle \in F_\xi$, we put $\langle \beta, i\rangle$ where we have to, just as in Subcase 1.1(b) and Subcase 1.2(b).

[Note that $C_\beta \subset B_\xi$ except in Subcase 1(a), where $C_\beta \cap B_\xi = \emptyset$.]

Case 2. If Case 1 does not hold, let $\mathfrak{B}_{\xi+1} = \mathfrak{B}_\xi$.

2. X is hereditarily screenable, collectionwise normal and hereditarily realcompact

We are now ready to finish showing all the advertised properties of (X, τ_w) except for the failure of countable metacompactness.

Proposition 2.1. *X is collectionwise normal.*

Proof. Let $\mathcal{F}$ be a discrete collection of closed sets. Let $\langle F^\rho \rangle_{\rho<\mathfrak{c}}$ list each nonempty member of $\mathcal{F}$ exactly once. [If necessary, $\emptyset$ gets listed $\mathfrak{c}$ times.] The closedness of each F^ρ and discreteness of $\mathcal{F}$ is witnessed by $\leq \mathfrak{c}$ subbasic open sets, while $\langle \mathcal{S}_\alpha : \alpha < 2^{\mathfrak{c}} \rangle$ is of cofinality $> \mathfrak{c}$. Thus, there is a first ξ such that $\mathcal{S}_\xi = \langle F^\rho \rangle_{\rho<\mathfrak{c}}$ and such that $\mathcal{F}$ is a discrete collection of closed sets in the topology generated by $\mathfrak{B}_\xi$. Then $\langle B^\rho_\xi \rangle_{\rho<\mathfrak{c}}$ is an open expansion of $\mathcal{F}$. $\square$

Proposition 2.2. *X is hereditarily realcompact.*

Proof. Because every point of X is a G_δ, it is enough to show that X is realcompact [8, Corollary 8.15].

Unlike the verification of these built-in properties, the proof that countable metacompactness fails is a long, arduous affair, as it is in all constructions of Dowker spaces by the Balogh technique. We need two long and technical sections before we can embark on that proof in earnest.

3. The failure of countable metacompactness, Part 1: Complete neighborhoods

Let $x \in X$. We introduce a notation for the basic neighborhoods of x, each a finite intersection of subbasic sets from $\mathfrak{B}$. Let H stand for the set of indices where we add something to the earlier subbases. That is, $H = \{\xi > 0 : \mathfrak{B}_{\xi+1} \neq \mathfrak{B}_\xi\}$. Then $H \cup \{0\} \supset \omega$. As in [6], we write H_1 for the Type I sequences in H, so that in our case $H_1 = H \setminus \omega$.

Definition 3.0. Let $x \in X$. A function t whose domain is a finite subset of H is called *compatible with* x if (a) $x \in B_\xi^{t(\xi)}$ (i.e., $t(\xi) = x(\xi)$) for all infinite ξ in the domain of t.

(b) if $n \in \omega \cap \mathrm{dom}(t)$ then $\emptyset \neq t(n) \in [\mathfrak{c}]^{<\omega}$ and $x \in \bigcap\{V^\rho_\xi : \rho \in t(\xi)\}$

If t is compatible with x, we let $B^t_\alpha = B^{t(\alpha)}_\alpha = B^{x(\alpha)}_\alpha$.

In the following section, we will choose a specific t for each $x \in X$ and label it $t(x)$. In this way, we will also look upon t as a function from X to $Fn(H_1, \mathfrak{c}) \cup Fn(\omega, [\mathfrak{c}]^{<\omega}$. [Recall that $Fn(I, J)$ stands for the set of functions whose domain is a finite subset of I, and whose range is a (finite) subset of J.] But by the end of that section, we will have settled on a specific x and will sometimes revert to the interpretation of t in Definition 3.0.

We next adopt a notation Balogh used in [B2], but with a slight difference in the basic open set it designates.

Notation 3.0.1. Let t be compatible with x. Let

$$V_{t,K,n}(x) = \bigcap_{\alpha \in \mathrm{dom}(t)} B^t_\alpha \cap \pi^{\leftarrow}(q_n) \cap D_x \setminus K.$$

where $\pi(x) \in q_n$ and K is a finite subset of $X \setminus \{x\}$.

The case $V_{\emptyset,K,n}(x) = \pi^{\leftarrow}(q_n) \cap D_x \setminus K$ will play an important role in the sequel. Clearly, the sets in 3.0.1 form an open neighborhood base for x. In fact, if the sets q_n are taken from the standard base for the Cantor set (thought of as 2^ω with the product topology) then these are precisely the finite intersections of the sets containing x in $\mathfrak{B}$.

[ASIDE: in [B2] there was an oversight here; Balogh left out W_{k+1} where $D_x (= W_k \cup \{x\})$ appears in the formula for $V_{t,K,n}(x)$ when $x \in L_k$, rendering a number of subsequent statements incorrect. The corrections will be noted below.]

Notation 3.0.2. For every infinite $\xi \in H$ let $O_\xi(x) = X \setminus F_\xi$ if $x \notin F_\xi$; while if $x \in F_\xi$, let $O_\xi(x) = (X \setminus F_\xi) \cup F_\xi^{\overline{x}(\xi)}$. [Recall that $\overline{x}(\xi)$ is the unique ordinal ρ such that $x \in F^\rho_\xi$.] Given t as in 3.0.0, and $\alpha \in \mathrm{dom}(t)$, let $U^t_\xi(x) = O_\xi(x)$ if ξ is infinite, and let $U^t_n(x) = D_{t(n),n}$ if n is finite. [In particular, if $x \in L_n$ then $U^t_n(x) = D_x$.]

Note that if t is compatible with x, then $x \in B^t_\alpha \subset U^t_\alpha(x)$. The sets $O_\xi(x)$ witness the relative discreteness of $\langle F^\rho_\xi : \rho < \mathfrak{c}\rangle$ in F_ξ and at the same time are open in X. The next concept entails witnessing the openness of the precursor of the next ingredient of $V_{t,K,n}(x)$, so to speak.

Definition 3.1. A basic open neighborhood $V_{t,K,n}(x)$ of x is *complete* if for every $\alpha \in \mathrm{dom}(t)$, $V_{t\restriction\alpha,K,n}(x) \subset U^t_\alpha(x)$.

Completeness Lemma 3.2. *Every point $x \in X$ has a neighborhood base consisting of complete neighborhoods.*

Proof. For an incomplete neighborhood $V_{t,K,n}(x)$, let $\alpha_{t,K,n}$ denote the greatest $\alpha \in \mathrm{dom}(t)$ such that $V_{t\restriction\alpha,K,n}(x)$ is not a subset of $U^t_\alpha(x)$. Our lemma follows from the well-ordering of the class of ordinals and the following:

Claim. *For an incomplete neighborhood $V_{t,K,n}(x)$, there is a neighborhood $V_{t',K',n'}(x)$ such that either $V_{t',K',n'}(x)$ is complete or $\alpha_{t',K',n'} < \alpha_{t,K,n}$*

Proof of claim.
[as in [B2]]

4. The failure of countable metacompactness, Part 2: reflecting appropriate β

The proof of the main result uses the following characterization of countable metacompactness, whose necessity follows very easily from the usual characterization of every countable open cover having a point finite refinement: any $\subset$-ascending family $\langle G_n \rangle_{m=0}^{\infty}$ of open sets covering X can be followed up by closed sets: there exist closed sets $Z_m \subset G_m$ such that $\bigcup_{m=0}^{\infty} Z_m = X$. We will apply this to the case $G_m = W_m$ and derive a contradiction at the end of Section 6.

Given a candidate sequence of Z_m's, we define ξ_m to be the unique ordinal in H for which $F^0_{\xi_m} = Z_m$, $F^1_{\xi_m} = X \setminus W_m$, and $F^\rho_{\xi_m} = \emptyset$ for all $\rho \geq 2$. In this way we get disjoint open sets $B^0_{\xi_m} \supset Z_m$ and $B^1_{\xi_m} \supset (X \setminus W_m)$. For every $x \in X \setminus W_m$, we define

$$V(x) = V_{t(x),K(x),n(x)}(x) \subset B^1_{\xi_m} \subset X \setminus Z_m$$

to satisfy $(T_0), (T_1)$, and (T_2) below. The first $\subset$ in the formula follows from the fact that $x \in X \setminus W_m$ and:

(T_0) $\{\xi_j : j \leq m\} \in \mathrm{dom}(t(x))$

[See 3.0.]

For the other two properties, we let $t_1(x) = \{\xi \in \mathrm{dom}(t) : \xi \geq \omega\}$. [Despite the similarity in notation, $t(x)$ is a function while $t_1(x)$ is a finite subset of H.] Given $C \subset X$, we let $t_1(C) = \bigcup_{x \in C} t_1(x)$. By passing to smaller and smaller neighborhoods if necessary, we ensure that the following hold:

(T_1) If $j < m < \omega$, $\xi \in t_1(\beta, j)$ and $\langle \beta, m \rangle \in B_\xi$, then $\xi \in t_1(\beta, m)$.

(T_2) Each $V(x) = V_{t(x),K(x),n(x)}(x)$ is a complete basic open neighborhood.

Now let M, N be countable elementary submodels of $H((2^{2^{\mathfrak{c}}})^+)$ (= the collection of all sets whose transitive closure has cardinality $\leq 2^{2^{\mathfrak{c}}}$) such that $M \in N$ and the following are members of M:

$$\mathfrak{c}, \quad \langle \mathcal{S}_\alpha \rangle_{\alpha < 2^{\mathfrak{c}}}, \quad \mathfrak{B}_\alpha \rangle_{\alpha < 2^{\mathfrak{c}}}, \quad H, \quad t : X \to Fn(H, \mathfrak{c}) \cup Fn(H, [\mathfrak{c}]^{<\omega},$$

$$t_1 : X \to [H_1]^{<\omega}, \quad \langle \xi_m \rangle_{m \in \omega}, \quad K : X \to [X]^\omega, \quad \langle x(\xi) : \xi \in H_1, x \in X \rangle$$

For the rest of this paper, we let $A = N \cap X$ $(= (N \cap \mathfrak{c}) \times \omega)$ and $\mathcal{D} = \{\mathcal{S}_\xi \restriction A : \xi \in M \cap H_1\}$. With A and $\mathcal{D}$ thus in hand, we next define the function u that we will be using in our control triples. One of its features is that $u(x)$ is finite for all $x \in \mathrm{dom}(u)$, even though other functions satisfying (C_3) and (C_4) in Definition 1.1 can have infinite $u(x)$.

Proposition 4.1. *There is a function u satisfying (C_3) and (C_4) in Definition 1.1, such that whenever $v : X \to [(H \setminus M) \cup \omega]^{<\omega}$ is an infinite partial function, $v \in N$ and $x \neq x'$ in A implies $v(x) \cap v(x') = \emptyset$, then there are infinitely many $x \in dom(v) \cap dom(u)$ such that*

$$u(x) = \{S_\xi \restriction A : \xi \in v(x)\}.$$

[Proof: as in [B2].]

For the rest of this section and the next two sections, we fix a $\beta \in \mathfrak{c}$ such that $\beta > \pi(A)$, $A_\beta = A$, $\mathcal{D}_\beta = \mathcal{D}$, and $u_\beta = u$.

Reflection Lemma 4.2. *Let $\theta \in M \cap H_1$, $k \in \omega$. Then there is a point $x = \langle \alpha, k \rangle \in dom(u) \subset N$ with the following properties:*

(R_0) $n(x) = n(\beta, k)$ [recall that this is the subscript on the Cantor set base member];

(R_1) $t_1(x) \cap M = t_1(\beta, k) \cap M$;

(R_2) $\langle \beta, k \rangle(\xi) \in M$ implies $\langle \beta, k \rangle(\xi) = x(\xi)$ for all $\xi \in t_1(\beta, k) \cap M$;

(R_3) $x \in B_\theta$ if and only if $\langle \beta, k \rangle \in B_\theta$; equivalently, if $\langle \beta, k \rangle$ is in one of the columns B_θ misses on account of Subcase 1(a)*, x is in one of those columns also, and vice versa;*

(R_4) If $\langle \beta, k \rangle \in B_\theta$, then either $\langle \beta, k \rangle \in M$ or $\langle \beta, k \rangle(\theta) \neq x(\theta)$;

(R_5) $x \in dom(u)$ and $u(x) = \{\mathcal{S}_\alpha \restriction A : \alpha \in \theta_1(x) \setminus M\}$.

Proof. We use the following notation: $n = n(\beta, k), r = t_1(\beta, k) \cap M, r^1 = \{\xi \in r : \langle \beta, k \rangle(\xi) \in M\}$ and $f(\xi) = \langle \beta, k \rangle(\xi)$ for every $\xi \in r^1$. Let $i = 1$ if $\langle \beta, k \rangle \in B_\theta$ and $i = 0$ if $\langle \beta, k \rangle \notin B_\theta$. Note that n, r, r^1, f, and i are in M. [In particular, i is simply either 0 or 1, depending on where $\langle \beta, k \rangle$ is situated.]

Let $\varphi(\alpha)$ denote the statement, "$n(\alpha, k) = n$ and $t_1(\alpha, k) \supset r$ and, for every $\xi \in r^1, \langle \alpha, k \rangle(\xi) = f(\xi)$ and $\langle \alpha, k \rangle \in B_\theta \iff i = 1$." All parameters in φ are in M, and $\varphi(\beta)$ is true. So, by elementarity of M, there are uncountably many α for which $\varphi(\alpha)$ is true. Elementarity of N then gives infinitely many α in N for which $\varphi(\alpha)$ is true. We will be using this and analogous results for analyzing the following two statements.

Let $\psi(E)$ denote the statement, "$E \subset \mathfrak{c}$ and $\forall \alpha \in E\ \varphi(\alpha)$, and $\alpha \neq \gamma \in E$ implies $[t_1(\alpha, k) \setminus r] \cap [t_1(\gamma, k) \setminus r] = \emptyset$."

Let $\chi(E)$ denote the statement, "$E \subset \mathfrak{c}$, and $\alpha \neq \gamma \in E$ implies $\langle \alpha, k \rangle, \langle \gamma, k \rangle \in B_\theta$ and $\langle \alpha, k \rangle(\theta) \neq \langle \gamma, k \rangle(\theta)$."

Claim.

(a) *There is an uncountable $E \in M$ such that $\psi(E)$ holds.*

(b) *Moreover, if $\langle \beta, k\rangle \in B_\theta$ and $\langle \beta, k\rangle(\theta) \notin M$, there is an uncountable $E \in M$ such that $\psi(E)$ and $\chi(E)$ both hold.*

$\vdash$ *Proof of Claim.* We prove (b) in detail; the proof of (a) is similar, and only requires removing all mention of $\chi(E)$ and the extra conditions on $\langle \beta, k\rangle$ in what follows.

By Zorn's lemma, there is a maximal E for which $\psi(E)$ and $\chi(E)$ both hold. Since all parameters in these two formulas are in M, we can (and will) assume that $E \in M$. Suppose that E is countable. Then, by elementarity, $E \subset M$. Let $E' = D \cup \{\beta\}$. Then E is a proper subset of E', but $\psi(E')$ holds, because $\forall \alpha \in E$,

$$[t_1(\alpha, k) \setminus r] \cap [t_1(\gamma, k) \setminus r] \ \subset \ t_1(\alpha, k) \cap [t_1(\beta, k) \setminus M] \ \subset \ M \cap [t_1(\beta, k) \setminus M] = \emptyset.$$

Also, $\chi(E')$ holds by the assumption in (b), due to (R_4) and the supposition that $\langle \alpha, k\rangle \in M$ for every $\alpha \in E$. But then E' contradicts the maximality of E. $\dashv$

Fix an E satisfying (a), and also (b) if $\langle \beta, k\rangle \in B_\theta$ and $\langle \beta, k\rangle(\theta) \notin M$. Let

$$E_1 = \{\alpha \in E : (t_1(\alpha, k) \setminus r) \cap M = \emptyset\}.$$

Clearly, $E_1 \in N$ and E_1 is a co-countable subset of E. Define a function v by setting $\mathrm{dom}(v) = E_1 \times \{k\}$ and $v(\alpha, k) = t_1(\alpha, k)) \setminus r$ for every $\alpha \in E_1$. Then $v \in N$ is as required in the statement of Proposition 4.1, so there are infinitely many $x = \langle \alpha, k\rangle \in \mathrm{dom}(u) \cap \mathrm{dom}(v)$ such that $u(x) = \{S_\xi \restriction A : \xi \in v(x)\}$. If $\langle \beta, k\rangle \in B_\theta$ and $\langle \beta, k\rangle(\theta) \notin M$, then $\chi(E)$ ensures that the $x(\theta)$'s for these x are all distinct. So, we can pick x from among these so that $\langle \beta, k\rangle(\theta) \neq x(\theta)$ in this case.

Such an $x = \langle \alpha, k\rangle$ then satisfies (R_4) and (R_5) by definition, $(R_0), (R_2)$, and (R_3) by $\varphi(\alpha)$, and (R_1) by $\varphi(\alpha)$ and the fact that $(t_1(\alpha, k)) \setminus r \cap M = \emptyset$. The cases where $\langle \beta, k\rangle \in B_\theta$ and $\langle \beta, k\rangle(\theta) \in M$, and where $\langle \beta, k\rangle \notin B_\theta$ are handled in the same way except that now (R_4) is satisfied automatically. $\square$

5. The failure of countable metacompactness, Part 3: homogeneity

The concept of homogeneity was the key to the rest of the proof that the spaces in [B1], [B2] and [Ny] are not countably metacompact. In our case it means the following.

Definition 5.0. *Let $\beta \in \mathfrak{c}$. We say β* **is ξ-homogeneous** *to mean*

$$(H) \text{ either } \xi \notin t_1(C_\beta) \text{ or there is } \gamma \in M \cap \mathfrak{c} \text{ such that } C_\beta \subset B^\gamma_\xi.$$

The β to which this will be applied are those in the last section: the ones such that $\beta > \pi(A)$ while $A_\beta = A, \mathcal{D}_\beta = \mathcal{D}$, and $u_\beta = u$.

If X were countably metacompact, we would have $\bigcup_{m\in\omega} Z_m = X$, where the sets Z_m are as at the beginning of Section 4. Thus it follows from the next two lemmas that X is not countably metacompact.

Lemma 5.1. *Let $\beta < \mathfrak{c}$. If $\bigcup_{m\in\omega} Z_m$ meets C_β, then there is an $m \in \omega$ such that β is not ξ_m-homogeneous.*

Proof. Fix m so that $Z_m \cap C_\beta \neq \emptyset$. Then by (T_0), $\xi \in t_1(\beta, m) \subset t_1(C_\beta)$. On the other hand, C_β is not contained in either $B^0_{\xi_m} (\supset Z_m)$ or $B^1_{\xi_m} (\supset X \setminus W_m)$. $\square$

Since $\xi_m \in M$ for all $m \in \omega$, we will be done once we show:

Main Lemma 5.2. *Let $\beta > \pi(A)$ satisfy $A_\beta = A, \mathcal{D}_\beta = \mathcal{D}$, and $u_\beta = u$. Then β is ξ-homogeneous for all $\xi \in M \cap H_1$.*

To begin the proof of the Main Lemma, we assume that it is false and consider the first $\theta \in M \cap [\omega, \mathfrak{c}) \cap H$ for which β is not θ-homogeneous. That is, $\theta \in C_\beta$ yet there is no γ in $M \cap \mathfrak{c}$ for which $C_\beta \subset B^\gamma_\theta$.

In what follows, let $y[k] = \{\langle \beta, j\rangle : j < k\}$. [Balogh had "$j \leq k$" instead of "$j < k$", because he was aiming for hereditary collectionwise normality; see Remark 2 below.]

Let k be big enough so that $\langle \beta, k\rangle \in B_\theta$ and:
(1_k) if $F_\theta \cap C_\beta \neq \emptyset$, then $F_\theta \cap y[k] \neq \emptyset$;
(2_k) if there are at least two $\rho \in \mathfrak{c}$ such that $F^\rho_\theta \cap C_\beta \neq \emptyset$, then there are at least two $\rho \in \mathfrak{c}$ such that $F^\rho_\theta \cap y[k] \neq \emptyset$;
(3_k) if there are at least two $\rho \in \mathfrak{c}$ such that $B^\rho_\theta \cap C_\beta \neq \emptyset$, then there are at least two $\rho \in \mathfrak{c}$ such that $B^\rho_\theta \cap y[k] \neq \emptyset$;
(4_k) $\theta \in t_1(\beta, k)$.
By (T_1) in Section 4, all large enough k satisfy (1_k) through (4_k).

Remark 2. Balogh's technique for ensuring hereditary collectionwise normality required going inside proper open subsets O_α as explained in Section 1. If there is a "ceiling" like L_n when $O_\alpha = W_{n+1}$, all of (1_k), (2_k), (3_k) could fail with the choice "$j < k$". For example, if $F^\rho_n = \{\langle \rho, n\}$ for all $\rho \in \mathfrak{c}$, then $F_n = L_n$ and $y[k] \cap F_n = \emptyset$. But without all three of these ℓ_k, the proof that X is not countably metacompact completely breaks down near the end.

For the rest of this section and the next, fix $x = \langle \alpha, k\rangle$ as in the Reflection Lemma for θ.

Lemma 5.2.1. *$y[k] \subset V_{\emptyset, K(x), n(x)}(x)$.*

Proof. Since $x \in N$, we also have $K(x) \in N$, and then $K(x) \subset N$ by finiteness of $K(x)$ and elementarity of N. Since $\beta \notin N$, it follows that $C_\beta \cap K(x) = \emptyset$. From (R_0) it follows that $y[k] \subset C_\beta \subset \pi^{\leftarrow}(q_{n(x)}$; also, $y[k] \subset W_k$, and we are done. $\square$

[ASIDE. Because [B2] omitted "$\cap W_k$" from the formula that defines $V_{t,K,n}(x)$, there was a hole in the proof at this point, but adding "$\cap W_k$" repairs it easily.]

Our choice of $j < k$ was dictated by the way x is all by itself in the top row of D_x: Lemma 5.2.1 would have been impossible for τ_w had Balogh's choice of $j \le k$ been adhered to. The same is true of:

Lemma 5.3. $y[k] \subset V_{t(x)\restriction\theta,K(x),n(x)}(x)$.

Proof. We will show that for every $\alpha \in \mathrm{dom}(t(x)) \cap \theta$,

$$(I_\alpha) \qquad y[k] \subset B_\alpha^{t(\alpha)}$$

holds. [The notation $B_\alpha^{t(\alpha)}$ reverts here to the original interpretation of t: it is what we have been calling $t(x)$ for some time.]

In the case of $\alpha = n \in \omega$, (I_α) is equivalent to

$$(I_n) \qquad y[k] \subset B_\alpha^t(x) = \bigcap_{\rho \in t(n)} V_n^\rho$$

and this is clear from the fact that $t(n) \subset A_\beta (= N)$ for all n, so that $\langle \beta, i\rangle \in V_n^\rho$ for all $i < n$ and $\rho \in t(n)$. [See the paragraph immediately following Definition 1.1.]

For infinite ξ, (I_ξ) is shown by induction. Suppose we have shown (I_η) for all η in $\mathrm{dom}(t(x)) \cap \alpha$. Then, by completeness of $V_{t(x),K(x),n(x)}(x)$,

$$(*) \qquad y[k] \subset V_{t(x)\restriction\alpha,K(x),n(x)}(x) \subset U_\alpha^t(x).$$

The first $\subset$ is by the induction hypothesis, the second by definition of "complete".

we consider two cases:

Case (a). $\xi \in H_1 \cap M$. Then by $\xi < \theta$ and minimality of θ it follows that β is ξ-homogeneous. Now $\xi \in \mathrm{dom}(t(x)) \cap H_1 \cap M$. By (R_1) in the Reflection lemma, $\xi \in t_1(\beta, k) \cap M \subset t_1(C_\beta)$. Hence by definition of ξ-homogeneity, there is $\gamma \in M \cap \mathfrak{c}$ such that $C_\beta \subset B_\xi^\gamma$. Making use of $(*)$, it follows that

$$y[k] \subset U_\xi^t \cap C_\beta = O_\xi \cap C_\beta \subset C_\beta \subset B_\xi^\gamma.$$

In particular, $\gamma = \langle \beta, k\rangle(\xi)$. By (R_2), $\gamma = x(\xi)$. Hence $y[k] \subset B_\xi^\gamma = B_\xi^{x(\xi)} = B_\xi^t(x)$.

Case (b). $\xi \in H_1 \setminus M$. Since $\xi \in \mathrm{dom}(t(x)) \cap \theta$, we get $\xi \in t_1(x) \setminus M$. So, by (R_5), $\mathcal{S}_\xi \restriction A_\beta \in u_\beta(x)$. By Subcase 2(b), to prove that $y[k] \subset B_\xi^{x(\xi)} = B_\xi^t(x)$, we need only show that $y[k] \subset O_\xi \setminus F_\xi) \cup F_\xi^{x(\xi)}$. This follows from the fact that by $(*)$, $y[k] \subset U_\xi^t(x) = O_\xi(x)$. [Recall that if $x \in O_\xi \setminus F_\xi$, then $O_\xi(x) = O_\xi \setminus F_\xi$, whereas $O_\xi(x) = O_\xi \setminus F_\xi \cup F_\xi^{\overline{x}(\xi)}$ if $x \in F_\xi$, and that in the latter case, $\overline{x}(\xi) = x(\xi)$.]

6. The failure of countable metacompactness, Part 4: the end of the proof of Main Lemma 5.2

To complete the proof that X is not countably metacompact, it remains to arrive at the promised contradiction:

Lemma 6.0. *β is θ-homogeneous.*

Proof. By (R_1) and (4_k), $\theta \in t_1(\beta, k) \cap M = t_1(x) \cap M$. Thus by Lemma 5.3 and completeness of $V_{t(x),K(x),n(x)}(x)$, we conclude that $y[k] \subset V_{t(x)\restriction\theta,K(x),n(x)}(x) \subset O_\theta(x)$.

Claim 1. $F_\theta \cap C_\beta \neq \emptyset$.

$\vdash$ Suppose that $F_\theta \cap C_\beta = \emptyset$. Because $\theta \in M \cap H_1 \setminus \omega$, $S_\theta \restriction A_\beta \in \mathcal{D}_\beta$. Hence by Subcase 1(a), it follows that $B_\theta \cap C_\beta = \emptyset$. On the other hand, it follows from $\theta \in t_1(\beta, k)$ that $\langle \beta, k\rangle \in B_\theta \cap C_\beta$, a contradiction. $\dashv$

Claim 2. There is precisely one $\gamma < \mathfrak{c}$ such that $F_\theta^\gamma \cap C_\beta \neq \emptyset$.

$\vdash$ By Claim 1, there is at least one such γ. Suppose there are more. Then by (2_k), there are at elast two γ such that $F_\theta \cap y[k] \neq \emptyset$. On the other hand, $y[k] \subset O_\theta(x)$, and $O_\theta(x)$ is either $O_\theta \setminus F_\theta$ or $(O_\theta \setminus F_\theta) \cup F_\theta^{\overline{x(\theta)}}$, a contradiction. $\dashv$

Next, let j be minimal with $\langle \beta, j\rangle \in F_\theta \cap C_\beta = F_\theta^\gamma \cap C_\beta$. Then, by (1_k), $\langle \beta, j\rangle \in y[k]$. In particular, $j < k$, and thus $y[k] \cap F_\theta \neq \emptyset$, and so $O_\theta(x) = (O_\theta \setminus F_\theta) \cup F_\theta^{\overline{x(\theta)}}$.

Claim 3. $x \in F_\theta^\gamma$.

$\vdash$ Since $\langle \beta, j\rangle \in F_\theta \cap y[k] \subset F_\theta \cap O_\beta(x)$, it follows that $\langle \beta, j\rangle \in F_\theta^{\overline{x(\theta)}}$. Since $\langle \beta, j\rangle \in F_\theta^\gamma$ by the definition of $\langle \beta, j\rangle$, it follows that $\gamma = \overline{x(\theta)}$, so $x \in F_\theta^\gamma$. $\dashv$

Claim 4. $y[k] \subset B_\theta^\gamma$.

$\vdash$ From $\langle \beta, j\rangle \in F_\theta^\gamma$, we got $\overline{\langle \beta, j\rangle} = \gamma = \overline{x(\theta)}$, and thus $y[k] \subset O_\theta(x) = (O_\theta \setminus F_\theta) \cup F_\theta^{\overline{\langle \beta, j\rangle}(\theta)}$. Hence, $y[k] \subset B_\theta^\gamma$ due to Subcase 1.1(b). $\dashv$

To complete the proof that β is θ-homogeneous, note that by (3_k), γ is the only ordinal ρ such that $B_\theta^\rho \cap C_\beta \neq \emptyset$. Furthermore, by the note just before Case 2 at the end of Section 1, we have $B_\theta \cap C_\beta = C_\beta$, whence $C_\beta \subset B_\theta^\gamma$. Thus,

$$\langle \beta, k\rangle(\theta) = \langle \beta, j\rangle(\theta) = \gamma = \overline{x(\theta)} = x(t).$$

Hence by (R_4), $\langle \beta, k\rangle(\theta) \in M$, and so β is θ-homogeneous, contrary to our assumption that θ is a counterexample. $\square$

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