A note on C_k(irrationals)

The main result in this note was obtained in 2001. It is that $C_k(\mathbb{P})$ does not have a σ-closure-preserving base at the origin consisting of countable unions of the usual basic open sets centered on the origin. A slightly different proof was subsequently published by Gartside and Glyn [1] so this note will not be published unless it is supplemented by new results.

Notation. In our context, X is a space homeomorphic to the space of irrational numbers with the usual topology, while $C_k(X)$ denotes the ring of continuous real-valued functions on X, with the compact-open topology. If $h \in C_k(X)$ and K is a compact subset of X and $\rho > 0$, let

$$B(h, K, \rho) = \{f \in X : |h(x) - f(x)| < \rho \text{ for all } x \in K\}.$$

As is well known, these sets form a base for $C_k(X)$.

Given a real number r, we let $\underline{r}$ denote the constant function with domain X and range $\{r\}$, and we let $\overrightarrow{r}$ denote the constant function with domain $C_k(X)$ and range $\{r\}$.

Let $\mathcal{T}$ denote the space of all nondecreasing functions from ω to itself, with the product topology. As is well known, $\mathcal{T}$ is homeomorphic to ${}^\omega\omega$ itself, and thus to the space $\mathbb{P}$ of irrationals. The following two facts are well known, but the proofs are so short that they are included here:

Theorem A. *$C_k(X)$ is a cosmic space; that is, it has a countable network.*

Proof. Let $\mathcal{B}$ be a countable base for $\mathbb{P}$, and for $B \in \mathcal{B}$ and rationals $q < r$, let $[B, (q, r)]$ = all f in $C_k(\mathbb{P})$ with $f(B) \subset (q, r)$, then the collection of all finite intersections of such things is a network. $\square$

Corollary. *$C_k(X)$ is hereditarily separable and hereditarily Lindelöf, and sequentially separable.*

Proof. A space is cosmic iff it is the continuous image of a separable metrizable space. Sequential separability is an immediate consequence, while the other two properties are easy consequences of having a countable network. $\square$

This corollary allows us to write any open set of $C_k(\mathbb{P})$ and hence of $C_\kappa(\mathcal{T})$ as a countable union of basic open sets. Because of the corollary and the following well-known theorem, the question of whether $C_k(\mathbb{P})$ is M_1 reduces to the question of whether there can be a σ-closure-preserving base of open sets about the origin $\overrightarrow{0}$.

Theorem B. [cf. [2], proof of Lemma 24] *If G is a topological group, and D is a dense subset of G, and $\mathcal{B}$ is an open base at the identity, then $\{dB : d \in D, B \in \mathcal{B}\}$ is a base for the topology on G.* $\square$

Theorem 1 below shows that if we have a base in $C_\kappa(\mathcal{T})$ at $\overrightarrow{0}$ of open sets which are the union of basic open sets as above, each of which is centered at $\overrightarrow{0}$, then the base cannot be σ-closure-preserving.

Lemma 1. *Given a base $\mathcal{B}$ at $\overrightarrow{0} \in \mathcal{T}$ of sets of the form*

$$\bigcup_{n=0}^{\infty} B(\overrightarrow{0}, K_n, \rho_n),$$

and $\rho > 0$, there are a family $\{B_\alpha : \alpha < \mathfrak{b}\} \subset \mathcal{B}$, and $p_\alpha \in \mathcal{T}$, and $\{q_n^\alpha : n \in \omega\} \subset \mathcal{T}$, and $\langle \rho_n^\alpha : n \in \omega \rangle$ with supremum $\rho_\alpha \leq \rho$ such that $\{p_\alpha : \alpha < \mathfrak{b}\}$ is $<^*$-increasing and $<^*$-unbounded, and such that

$$B(\overrightarrow{0}, (q_n^\alpha)^\downarrow, \rho_n^\alpha) \ \subset \ B_\alpha \ \subset \ B(\overrightarrow{0}, p_\alpha^\downarrow, \rho_\alpha)$$

for all $n \in \omega$.

Proof. Choose the p_α first, and make an initial choice of ρ'_α. Since $\mathcal{B}$ is a base at $\overrightarrow{0}$, we can assume that if $B \in \mathcal{B}$, then all ρ_n associated with B as in the statment of this lemma are less than ρ; and since each $B(\overrightarrow{0}, p_\alpha^\downarrow, \rho'_\alpha)$ contains some member of the base, we can choose $B_\alpha \in \mathcal{B}$ to be contained in it. If

$$B_\alpha = \bigcup_{n=0}^{\infty} B(\overrightarrow{0}, K_n^\alpha, \rho_n^\alpha),$$

let ρ_α be the supremum of the ρ_n^α; clearly $\rho_\alpha \leq \rho'_\alpha$, and since each $B(\overrightarrow{0}, K_n^\alpha, \rho_n^\alpha)$ is a subset of $B(\overrightarrow{0}, p_\alpha^\downarrow, \rho'_\alpha)$, and $\rho_n^\alpha \leq \rho_\alpha$, we have $K_n^\alpha \subset p_\alpha^\downarrow$ and so $B(\overrightarrow{0}, K_n^\alpha, \rho_n^\alpha) \subset B(\overrightarrow{0}, p_\alpha^\downarrow, \rho_\alpha)$ for all n. Finally, it is clear that if q_n^α majorizes the compact set K_n, then $B(\overrightarrow{0}, (q_n^\alpha)^\downarrow, \rho_n^\alpha) \subset B_\alpha$ for all n. $\square$

Theorem 1. *If $\mathcal{B}$ is a base at $\overrightarrow{0} \in \mathcal{T}$ of sets of the form*

$$\bigcup_{n=0}^{\infty} B(\overrightarrow{0}, K_n, \rho_n),$$

then $\mathcal{B}$ is not σ-closure-preserving.

Proof. Let p_α, B_α, etc. be as in Lemma 1. Taking a subfamily of $\mathfrak{b}$ members if necessary, we may assume $\{\rho_\alpha : \alpha < \mathfrak{b}\}$ is both bounded in $\mathbb{R}$ and bounded away from 0. By the same kind of cutting-down-if-necessary argument, it is enough to show that $\{B_\alpha : \alpha < \mathfrak{b}\}$ is not closure-preserving.

Let $q_\alpha = q_0^\alpha$ for all α. Then $p_\alpha(k) \leq q_\alpha(k)$ for all $k \in \omega$, because of the double containment in the statement of Lemma 1.

Inductively define $q \in \mathcal{T}$ one coordinate at a time so that, for all $n \in \omega$, there are $\mathfrak{b}$-many q_α extending $q \restriction n$. The associated p_α's are still $<^*$-unbounded, so that after q has been defined, there will be an infinite set of integers k for which the following set is unbounded in ω:

$$\{p_\alpha(k) : q_\alpha \restriction k = q \restriction k\}$$

Therefore, we can define a strictly increasing sequence of ordinals $\langle \alpha_n \rangle_{n\in\omega}$ by induction, along with a strictly increasing sequence of non-negative integers k_n, so that $p_{\alpha_0}(k_0) > q(k_0)$ and, if $n > 0$:

(1) $q_{\alpha_n}(j) = q(j)$ for all $j \leq k_{n-1}$;
(2) $p_{\alpha_n}(k_n) > q(k_n)$; and
(3) $p_{\alpha_n}(k_n) > q_{\alpha_i}(k_n)$ for all $i < n$.

Notation: For $\sigma \in {}^i\omega$, let $U[\sigma] = \{p \in \mathcal{T} : p \restriction i = \sigma\}$

Let $\sigma_n = p_{\alpha_n} \restriction k_n + 1$. From (1) – (3) and the fact that $p_\alpha \leq q_\alpha$ it follows that $p_{\alpha_n}(k_n) > q_{\alpha_m}(k_n)$ for all $m \neq n$. From this follows item (b) below, and part (c) follows similarly:

(a) $p_{\alpha_n} \in U[\sigma_n]$;
(b) $U[\sigma_n] \cap q_{\alpha_m}^{\downarrow} = \emptyset$ for all $m \neq n$; and
(c) $U[\sigma_n] \cap U[\sigma_m] = \emptyset$ for all $m \neq n$,

Another easy consequence of (1) and the definition of σ_n and the fact that $p_\alpha \leq q_\alpha$ is that the boundary of $\bigcup_{n\in\omega} U[\sigma_n]$ is a (compact) subset of $q^{\downarrow}$.

Remarkably enough, although the q_n^α with $n > 0$ play no role in these definitions, there is a subsequence of $\langle B_{\alpha_n} : n \in \omega\rangle$ and a function h in the closure of the union of the members of the subsequence, but not in the closures of the individual members. These other q_n^α come into play in defining h, via a concept of "dangerous for n wrt j" introduced below. Once we find the desired subsequence, Theorem 4 follows.

By picking a subsequence if necessary, we may assume $\rho_{\alpha_n} \to \rho > 0$. The remainder of the proof is covered by two main cases. Case 1 is where $\rho_{\alpha_n} \uparrow \rho$, while Case 2 is where $\rho_{\alpha_n} \searrow \rho$, where by $r_n \uparrow r$ is meant that $r_n \to r$ and $r_n < r_{n+1}$ for all n, whereas by $r_n \searrow r$ is meant that $r_n \to r$ and $r_n \geq r_{n+1}$ for all n. Clearly every convergent sequence of reals has a subsequence falling under one of these two descriptions.

Case 1. Let $h = \underline{\rho}$. This is the easy case; h is in the uniform closure of the union of the B_{α_n} but is not even in the product-topology closure of the individual B_{α_n}. Indeed $B(h, \{p_{\alpha_n}\}, \rho - \rho_{\alpha_n})$ misses even $B(\overrightarrow{0}, p^{\downarrow}_{\alpha_n}, \rho_{\alpha_n})$. On the other hand, if $\epsilon > 0$, pick $N \in \omega$ so that $\rho - \rho_{\alpha_N} < \epsilon$, pick k so that $\rho - \rho^{\alpha_N}_k < \epsilon$, and let $g(x) = \rho^{\alpha_N}_k - \delta$ for all $x \in \mathcal{T}$, where $\delta < \epsilon - \rho + \rho^{\alpha_N}_k$. Then clearly $g \in B(\overrightarrow{0}, (q^{\alpha_N}_k)^{\downarrow}, \rho^{\alpha_N}_k)$, while $||h - g||_\infty = \rho - \rho^{\alpha_N}_k < \epsilon$, so *a fortiori* $g \in B(h, K, \epsilon)$ for all compact K.

Case 2. In this case we may find ourselves taking subsequences infinitely many times, so to avoid too many multiple subscripts and a confusing tangle of "wolog"s, we will index the sequences using a decreasing chain of subsets of ω. The final outcome will be a subset $A_\omega = \{n(j) : j \in \omega\}$ of ω and a function h in the closure of $\bigcup\{B_{\alpha_n} : n \in A_\omega\}$ but not in the closure of any individual B_{α_n} $(n \in A_\omega)$. We define $h \restriction U[\sigma_n](n \in A_\omega)$ by induction, beginning with the assumption that $\rho_{\alpha_n} \searrow \rho\,(> 0)$.

Let $n(0) = 0$, let $A_0 = \omega$, let $\varepsilon_0 = \rho_{\alpha_0} + 1$, and let $h(x) = \varepsilon_0$ for all $x \in U[\sigma_0]$. Given $n \in \omega \setminus \{0\}$ and $r \in \mathbb{R}$, call r *dangerous for n wrt* 0 if, for each compact $K \subset \mathcal{T}$, there exists k such that $K \cap K^{\alpha_n}_k \cap U[\sigma_0] = \emptyset$, and such that $\rho^{\alpha_n}_k \geq r$. From (b) above, it follows that $\rho^{\alpha_n}_0$ is dangerous for all $n > 0$ wrt 0, while it is clear that no $r > \rho_{\alpha_n}$ is dangerous for any $n > 0$ wrt 0.

For each $n > 0$, let $\delta^0_n = \sup\{r : r \text{ is dangerous for } n \text{ wrt } 0\}$. If $\langle \delta^0_n \rangle$ has a strictly increasing (convergent) subsequence, $\langle \delta^0_n : n \in A\rangle \uparrow \delta_0$ $(\leq \rho)$, let $A_\omega = \omega$, and let $h(x) = \delta_0$ whenever $x \notin U[\sigma_0]$. On the other hand, if there is no such subsequence, then there is a monotone non-increasing subsequence $\langle \delta^0_n : n \in A_1\rangle$ converging to some δ_0 $(\leq \rho)$. In this case let $n(1) = \min(A_1)$, let $\varepsilon_{n(1)} = \delta^0_{n(1)} + 1/2$, and let $h(x) = \varepsilon_{n(1)}$ for all $x \in U[\sigma_{n(1)}]$. Note that $\varepsilon_{n(1)}$ is not dangerous for any $n \in A_1$ wrt 0. Continue building h by induction as follows.

The general induction hypothesis at $j \in \omega \setminus \{0\}$ is that A_i and $n(i) = min(A_i)$ and $h \restriction U[\sigma_{n(i)}]$ have been defined for all $i \leq j$ and that:

$$(*) \qquad \langle \delta^i_n : n \in A_i\rangle \searrow \delta_i \text{ for all } i \leq j, \text{ and if } m < i < j \text{ then } \rho \geq \delta_m \geq \delta_i > 0.$$

For each $n \in A'_j (= A_j \setminus \{\min A_j\})$ and each $r \in \mathbb{R}$, call r *dangerous for n wrt j* if the following holds:

For each compact K there exists k such that $K \cap K^{\alpha_n}_k \cap \bigcup_{i \leq j} U[\sigma_{n(i)}] = \emptyset$, and such that $\rho^{\alpha_n}_k \geq r$.

Again by (b), $\rho_0^{\alpha_n}$ is dangerous for all $n \in A'_j$ wrt j. It is also easy to see that if r is dangerous for n wrt j, then r is dangerous for n wrt i for all $i < j$; and also that the set of reals dangerous for n wrt j forms an initial segment of $\mathbb{R}$.

Obviously, r is *not* dangerous for n wrt j iff there exists a compact subset K of $\mathcal{T}$ such that for each $k \in \omega$, either $K \cap K_k^{\alpha_n} \cap \bigcup_{i \leq j} U[\sigma_{n(i)}] \neq \emptyset$, or $\rho_k^{\alpha_n} < r$.

The purpose of this concept is to find a value for $h(x)$ on $U[\sigma_n]$ which will put h outside the closure of B_{α_n}, but which also makes it possible to continue defining h on the rest of $\mathcal{T}$ so that it will be in the closure of the union of all the B_{α_m}. If r is dangerous for n wrt j, and if $h \leq r$ on $K \cap K_k^{\alpha_n}$ whenever this intersection is nonempty, then h is in the closure of B_{α_n} no matter how h is defined elsewhere.

For each $n \in A'_j$ let $\delta_n^j = \sup\{r : r \text{ is dangerous for } n \text{ wrt } j\}$. If $\langle \delta_n^j : n \in A'_j \rangle$ has a strictly increasing subsequence, $\langle \delta_n^j : n \in A_{j+1} \rangle$, let its limit be $\delta_j (\leq \delta_{j-1})$; let $A_\omega = A_{j+1} \cup \{n(i) : i \leq j\}$ and let $h(x) = \delta_j$ for all $x \notin \bigcup_{i \leq j} U[\sigma_{n(i)}]$ Clearly, h is continuous. For notational convenience, write $\varepsilon_m = \delta_j$ for all $m > n(j), m \in A_\omega$, and have $\{n(j) : j \in \omega\}$ list A_ω in its natural order. By definition of δ_j, ε_m is not dangerous for n wrt j for any $n \in A_{j+1}$.

On the other hand, if $\langle \delta_n^j : n \in A'_j \rangle$ has no strictly increasing subsequence, then there is a monotone non-increasing subsequence $\langle \delta_n^j : n \in A_j \rangle$ converging to some $\delta_j (\leq \delta_{j-1})$. In this case let $n(j+1) = \min(A_j)$, let $\varepsilon_{n(j+1)} = \delta_{n(j+1)}^j + 1/2^{j+1}$, and continue the induction. It is clear from the definition of δ_n^j that $\varepsilon_{n(j+1)}$ is not dangerous for any $n \in A_{j+1}$ wrt j.

If the induction is forced to continue for infinitely many steps, let $A_\omega = \{n(i) : i \in \omega\}$. Then $\{\varepsilon_m : m \in A_\omega\}$ is a monotone non-increasing sequence converging to some $\delta \leq \rho$. We then define h on $U[\sigma_{n(j}]$ to equal $\varepsilon_{n(j)}$ and let h equal δ everywhere outside $\bigcup\{U[\sigma_m] : m \in A_\omega\}$. Then h is clearly continuous.

$\vdash$ *If* $m \in A_\omega$*, then* h *is not in the closure of* B_{α_m}. This is clear in case $m = 0$, since then $B(h, \{p_{\alpha_0}\}, 1)$ even misses $B(\overrightarrow{0}, p_{\alpha_0}^{\downarrow}, \rho_{\alpha_0})$. Otherwise, we have $\varepsilon_m = \varepsilon_{n(j+1)}$ for some j, and no $r > \delta_m^j$ is dangerous for m wrt j, so there is a compact set C that meets every set of the form $K_k^{\alpha_m} \cap \bigcup_{i \leq j} U[\sigma_{n(i)}]$ for which $\rho_k^{\alpha_m} > \delta_m^j$. Let $K = \{p_{\alpha_m}\} \cup C$.

Suppose first that $A_\omega = A_{j+1} \cup \{n(i) : i \leq j\}$ for some j, and that $m \in A_{j+1}$, so that $\varepsilon_m = \delta_j$. Let $\epsilon = \min\{\delta_j - \delta_m^j, 1/2^j\}$.

$\vdash\vdash$ *Claim.* $B(h, K, \epsilon)$ *does not meet* $B_{\alpha_m} = \bigcup_{k=0}^{\infty} B(\overrightarrow{0}, K_k^{\alpha_m}, \rho_k^{\alpha_m})$.

Proof of Claim. Fix $k \in \omega$. If $K \cap K_k^{\alpha_m} \cap U[\sigma_{n(i)}] = \emptyset$ for all $i \leq j$, and $f \in B(\overrightarrow{0}, K_k^{\alpha_m}, \rho_k^{\alpha_m})$, then $f(p_{\alpha_m}) < \delta_m^j$, whereas $h(p_{\alpha_m}) = \delta_j$.

On the other hand, if $K \cap K_k^{\alpha_m} \cap \bigcup_{i \leq \ell} U[\sigma_{n(i)}] \neq \emptyset$, let $p \in K \cap K_k^{\alpha_m} \cap U[\sigma_{n(i)}]$ for the least $i \leq j$ for which this is possible. If $i = 0$ then $f(p) < \rho_{\alpha_0}$ whereas $h(p) = \rho_{\alpha_0} + 1$. If $i > 0$, then, by minimality of i, and by the fact that any $r > \delta_{n(i)}^{i-1}$ is not dangerous for m with respect to $i-1$, we have $f(p) < \delta_{n(i)}^{i-1}$ for all $f \in B(\overrightarrow{0}, K_k^{\alpha_m}, \rho_k^{\alpha_m})$, whereas $h(p) = \delta_{n(i)}^{i-1} + 1/2^i \geq \delta_{n(i)}^{i-1} + 1/2^j$. ⊣⊣

If the induction continues to where A_{j+2} is defined, then we let $\epsilon = 1/2^{j+2}$ and follow the above argument, except that now $h(p_{\alpha_m}) = \delta_m^j + 1/2^{j+1}$. ⊣

⊢ *h is in the closure of $\bigcup\{B_{\alpha_m} : m \in A_\omega\}$.* Given $\epsilon > 0$ and a compact subset K of $\mathcal{T}$, choose j in ω as follows. If the induction stops at some stage j, pick $\ell > j$ large enough so that $\delta_{n(\ell)}^j > \delta_j - \epsilon/2$, while if the induction continues for infinitely many steps, choose ℓ so that $\varepsilon_{n(\ell)} - \delta < \epsilon/2$. In the latter case, let $m = n(\ell)$; then $\delta_{n(\ell)}^\ell - \epsilon/2$ is dangerous for m wrt $\ell - 1$. Hence there exists k such that

$$K_k^{\alpha_m} \cap \bigcup_{i<\ell} U[\sigma_{n(i)}] \cap K = \emptyset,$$

and such that $\rho_k^{\alpha_m} \geq \delta_{n(\ell)}^{\ell-1} - \epsilon/2$. This enables us to choose $g \in \overline{B(\overrightarrow{0}, K_k^{\alpha_m}, \rho_k^{\alpha_m})} \cap B(h, K, \epsilon)$ as follows. Let g be any continuous function which agrees with h on $K \cap \bigcup_{i \leq \ell} U[\sigma_{n(i)}]$ and equals $\rho_k^{\alpha_m}$ on $K_k^{\alpha_m}$. Then g is as desired: if $p \in K$ then $g(p)$ is within ϵ of $h(p)$, etc.

The former case, where the induction stops at stage $j < \ell$, is similar: we have that $\delta_{n(\ell)}^j - \epsilon/2$ is dangerous for $m = n(\ell)$ wrt j, and now we look for a k so that

$$K_k^{\alpha_m} \cap \bigcup_{i \leq j} U[\sigma_{n(i)}] \cap K = \emptyset,$$

and such that $\rho_k^{\alpha_m} \geq \delta_{n(\ell)}^j - \epsilon/2$, etc. ⊣ □

References

[1] P. Gartside and A. Glyn, "Closure preserving properties of C_k (metric fan)," Topology Appl. 151 (2005), no. 1-3, 120–131.

[2] P.M. Gartside and E.A. Reznichenko, "Near metric properties of function spaces," *Fund. Math.* 164 (2000) 97–114.