UPDATES ON A 1903 THEOREM OF G.H. HARDY

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[Incomplete Draft]

1. The axiom of G.H. Hardy, equivalents and status

In the year before Zermelo published his proof of the well-ordering principle from AC, the renowned Cambridge mathematician G. H. Hardy published a proof that there is an uncountable well-orderable subset of the real line [1], [2]. Hardy's technique was surprisingly modern: he used a ladder system on ω_1 to build an uncountable set of sequences of natural numbers well-ordered by the preorder $<^*$ of eventual domination. We now know that some form of the axiom of choice (AC) is needed for this, and even that the existence of Hardy's uncountable set does not imply the existence of a ladder system on ω_1 just assuming ZF.

Definition 1. Given a limit ordinal α, a *ladder at* α is a strictly ascending sequence of ordinals less than α whose supremum is α. Given an ordinal γ, a *ladder system on* γ is a family

$$\{L_\alpha : \alpha \in \gamma,\ \alpha \text{ is a limit ordinal of countable cofinality}\}$$

where each L_α is a ladder at α.

Theorem 1. *In ZF, the following axioms are equivalent:*

1. *There is a ladder system on* ω_1.

2. *There is a Hausdorff gap.*

3a. *There is a well-orderable special Aronszajn tree.*

3b. *There is a well-orderable special tree of height* ω_1.

3c. *There is an* $\mathbb{R}$*-special tree* T *of height* ω_1 *with a choice function for the levels of* T.

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4a(i). *There is a coherent family* $\{f_\alpha : \alpha \in \omega_1\}$ *of 1-1 functions* $f_\alpha : \alpha \to \omega$.

4a(ii). *There is a family* $\{f_\alpha : \alpha \in \omega_1\}$ *of 1-1 functions* $f_\alpha : \alpha \to \omega$.

4a(iii). *Given any disjoint family* $\mathcal{A}$ *of countably infinite well-ordered sets, there is a function* $F : \bigcup \mathcal{A} \to \omega$ *such that the restriction of* F *to any member of* A *is a bijection.*

4b. *There is a family* $\{h_\alpha : \alpha \in \omega_1\}$ *of surjective functions* $h_\alpha : \omega \to \alpha$.

4c(i). *The direct sum of the countable ordinals is special when viewed as a tree.*

4c(ii). *The direct sum of any family of trees of countable height is special.*

5a. *The topological direct sum of any family of countable ordinal spaces is metrizable.*

5b. *The topological direct sum of any family of countable ordinal spaces can be given a uniformity with a countable base of equivalence relations.*

6a. *The topological direct sum of any family of countable locally compact metrizable spaces is the countable union of discrete subspaces.*

6b. *The topological direct sum of any family of scattered Hausdorff spaces of countable Cantor-Bendixson rank is the countable union of discrete subspaces.*

7a. *The open first octant in* $\omega_1 \times \omega_1$ *is a* β*-space.*

7b. *The open first octant in* $\omega_1 \times \omega_1$ *is an elementary* β*-space.*

Definitions.

2. A *Hausdorff gap* is a pair $(\mathcal{A}, \mathcal{B})$ where $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ and $\mathcal{B} = \{B_\alpha : \alpha < \omega_1\}$ are families of subsets of ω such that

(1) $A_\alpha \subset^* A_\beta \subset^* B_\beta \subset^* B_\alpha$ whenever $\alpha < \beta$ [note order reversal in third equation] and
(2) For all $\beta < \omega_1$, $\{A_\alpha : \alpha < \beta\}$ leaks badly out of B_β, meaning that for each integer n there are only finitely many $\alpha < \beta$ such that $A_\alpha \setminus B_\beta$ is a subset of $\{0, \dots, n\}$.

Here $\subset^*$ is the strict pre-order of almost containment: $A \subset^* B$ means that $A \setminus B$ is finite and $B \setminus A$ is infinite. [Some authors omit the second condition.]

Every Hausdorff gap is an (ω_1, ω_1^*)-gap: a pair $(\mathcal{A}, \mathcal{B})$ satisfying (1) and (2^-): there is no subset C of ω_1 satisfying $A_\alpha \subset^* C \subset^* B_\alpha$ for all $\alpha \in \omega_1$.

3. A *tree* is a partially ordered set in which the predecessors of any element are well-ordered. [Given two elements $x < y$ of a poset, we say x is a *predecessor* of y and y is a *successor* of x.] If T is a tree, then $T(0)$ is its set of minimal members. Given an ordinal α, if $T(\beta)$ has been defined for all $\beta < \alpha$, then $T \restriction \alpha = \bigcup\{T(\beta) : \beta < \alpha\}$, while $T(\alpha)$ is the set of minimal members of $T \setminus T \restriction \alpha$. The set $T(\alpha)$ is called *the* α*-th level of* T.

A tree is *Aronszajn* if it is uncountable while every chain is countable and every level $T(\alpha)$ is countable. A tree is *special* if it is a countable union of antichains. A tree T is $\mathbb{R}$-*special* if there is a strictly order-preserving function $f : T \to \mathbb{R}$, *i.e.,* $s < t$ in T implies $f(s) < f(t)$.

Caution! Some authors require Aronszajn trees to satisfy the additional requirement that every element has uncountably many successors—equivalently, successors on every higher level of the tree. Some even require that every element has at least two immediate successors. This makes little difference in ZFC because then any Aronszajn tree can be pruned to produce one satisfying the extra requirements. However, there are models of ZF where some trees satisfying Definition 4 have no subtrees satisfying either of the extra requirements.

4. A family of functions is called *coherent* if any two functions agree on all but finitely many members of their common domains.

The *direct sum* of ordered sets $\{(A_\gamma, <_\gamma) : \gamma \in \Gamma\}$ is $\bigcup\{A_\gamma \times \{\gamma\} : \gamma \in \Gamma\}$ with the order $(a, \gamma_1) < (b, \gamma_2)$ iff $\gamma_1 = \gamma_2$ and $a <_{\gamma_1} b$.

5. The *direct sum* of topological spaces $\{(X_\gamma, \tau_\gamma) : \gamma \in \Gamma\}$ is $\bigcup\{X_\gamma \times \{\gamma\} : \gamma \in \Gamma\}$ with the topology whose base is all sets of the form $\bigcup\{U_\gamma \times \{\gamma\} : \gamma \in \Gamma, U_\gamma \in \tau_\gamma\}$.

6. A topological space X is *scattered* if every subspace has an isolated point in the relative topology. The *Cantor-Bendixson rank* of a point $x \in X$ is defined by induction similarly to the height of a point in a tree: $R(0)$ is the set of isolated points of X; if $R(\beta)$ has been defined for all $\beta < \alpha$, then $R(\alpha)$ is the set of isolated points of the subspace $X \setminus \bigcup\{R(\beta) : \beta < \alpha\}$. The Cantor-Bendixson *rank* of a scattered space is the least α such that $R(\alpha)$ is empty.

7. A topological space X is a β-*space* if there is a family of open sets $\{g(n, x) : n \in \omega, x \in X\}$ of such that $g(n, x)$ is a neighborhood of x for all n and x, and such that $\langle x_n$ has a cluster point whenever $\bigcap_{n=0}^{\infty} g(n, x_n) \neq \emptyset$. A β-space is *elementary* if $g(n, x) = g(0, x)$ for all n and all x.

The axiom 1. in Theorem 1 was designated "Postulate A" in [3] and the implication 1. $\Longrightarrow$ 4a(ii). is Corollary 1 of Theorem A_1 in [3]; the converse is easy. The implication 1. $\Longrightarrow$ 2 is essentially proven in [4], and the converse is trivial.

"Postulate A" is an easy consequence of the following slight weakening of the axiom of choice, which is actually equivalent to it in ZF but is strictly weaker if atoms (Urelemente) are allowed:

MC(∞, ∞): The *Multiple Choice Axiom*: For every family $\mathcal{A} = \{A_\gamma : \gamma \in \Gamma\}$ of non-empty pairwise disjoint sets there exists a family $\mathcal{F} = \{F_\gamma : \gamma \in \Gamma\}$ of finite non-empty sets such that for every $\gamma \in \Gamma$, $F_\gamma \subseteq A_\gamma$.

Of course, it is enough to have the special case MC$(\aleph_1, 2^{\aleph_0})$ of MC(∞, ∞) which has $|\Gamma| \leq \aleph_1$ and $|A_\gamma| \leq 2^{\aleph_0}$ for all $\gamma \in \Gamma$. It would be interesting to know of interpolants between MC$(\aleph_1, 2^{\aleph_0})$ and "Postulate A".

In Theorem 1, item 3c, "choice function" can be weakened to "multiple choice function," *i.e.,* a function that picks a finite subset of each level of the tree. This is partly because the union of a finite collection of ranges of ladders is the range of a ladder.

Theorem 2. [G.H. Hardy] [1] *"Postulate A" implies the existence of an uncountable well-orderable subset of* $\mathbb{R}$.

Hardy took advantage of the fact (which can be proven easily in ZF) that there is an injective function from the family of sequences $\sigma : \omega \to \omega$ of natural numbers into $\mathbb{R}$. He showed, in effect:

Theorem 3. *"Postulate A" implies that there is a family* $\{\sigma_\alpha : \alpha < \omega_1\}$ *of sequences of natural numbers such that* $\sigma_\alpha <^* \sigma_\beta$ *whenever* $\alpha < \beta$.

Here $<^*$ is the order of eventual domination: $f <^* g$ means there exists $n \in \omega$ such that $f(i) < g(i)$ for all $i > n$.

Theorem 4. *"Postulate A" implies that the long line is smoothable.*

The problem of obtaining "Postulate A" without relying on any variants of the axiom of choice was posed in [2].

Postulate A was treated at length in [3] along with two alternatives, the three together being mutually exclusive and mutually exhaustive:

Postulate B: Postulate A fails, but for each limit ordinal "of the second class" there is a ladder system on that ordinal.

Postulate C: There is a limit ordinal "of the second class" which does not admit of a ladder system.

Under Postulates A and B, the "ordinal numbers of the second class" coincide with the countable infinite ordinals. Under Postulate C, they contain the countable infinite ordinals as a proper subset. In [3] there is a list of axioms, reminiscent of the Peano Axioms, for the "ordinal numbers of the second class."

Also in [3] it is shown, in effect, that "Postulate C" is equivalent to ω_1 being of countable cofinality, and hence that "Postulate A" and "Postulate B" both imply ω_1 is regular; and that both "Postulate B" and "Postulate C" imply $\mathbb{R}$ cannot be well-ordered. As a corollary, we then have:

Theorem 5. *If* $\mathbb{R}$ *can be well-ordered, there is a ladder system on* ω_1. □

Church [3] left unanswered the natural question whether either "Postulate B" or "Postulate C" is compatible with $\mathbb{R}$ having an uncountable well-ordered subset. The answer to both parts of this question is affirmative. Assuming the consistency of an inaccessible cardinal, Howard Becker has recently produced a model in which the axiom of Dependent Choices holds (hence ω_1 is regular), there is no ladder system on ω_1 (hence we have a model of Postulate B), and there is an uncountable well-orderable subset of $\mathbb{R}$.

For the other part of the question, one can use a method of Feferman and Lévy which produces models of ZF in which ω_1 is singular. The method introduces surjective maps $f_n : \omega \to \omega_n$ into a symmetric submodel $\mathfrak{N}$ of a forcing model $\mathfrak{M}[G]$ without introducing the set of these maps, nor even a map from ω onto ω_ω. [See Jech's *Set Theory*, pp. 213–4 for the details.] Thus in the new model $\mathfrak{N}$, $\aleph_1$ is $\aleph_\omega$ of the ground model $\mathfrak{M}$, and the sequence of ground model $\aleph_n$'s becomes a countable cofinal subset of $\omega_1^{\mathfrak{N}}$. Now if the ground model $\mathfrak{M}$ satisfies MA + $\mathfrak{c} = \aleph_{\omega+1}$, then $\mathfrak{M}$ contains an $\omega_{\omega+1}$-scale: a sequence of functions $g_\alpha : \omega \to \omega$, $(\alpha < \omega_{\omega+1})$ which is well ordered by the order $<^*$ and which is cofinal in $({}^\omega\omega, <^*)$. In the symmetric extension $\mathfrak{N}$, this sequence loses its $<^*$-cofinality but remains $<^*$-unbounded, and it can easily be used to produce a subset of $\mathbb{R}$ itself which is well-orderable in order type $\omega_2^{\mathfrak{N}} = \omega_{\omega+1}^{\mathfrak{M}}$.

Andreas Blass, adapting an old argument of Solovay, has shown that if "Postulate B" [equivalently: ω_1 is regular but there is no ladder system on ω_1] holds, then ω_1 is inaccessible in Gödel's Constructible Universe L. Hence the inaccessible is necessary in Becker's model.

It is easy to see that the existence of an (ω_1, ω_1^*)-gap implies both (a) that there is an uncountable set of sequences well-ordered by $<^*$ and hence an uncountable well-orderable subset of $\mathbb{R}$ and (b) that ω_1 is not of countable cofinality—equivalently, ω_1 is regular—and so a negative answer to the following problem would also require an inaccessible cardinal:

Problem 1. Does the existence of an (ω_1, ω_1^*)-gap imply the existence of a Hausdorff gap?

2. Superficially similar but weaker axioms

Here are some consequences of "Postulate A" which are superficially similar to some of the statements in Theorem 1, but are strictly weaker.

(I) There exists a system of countable families of functions, $\mathcal{F} = \{F_\alpha : \alpha < \omega_1\}$, such that if $f \in F_\alpha$, then f maps α one-to-one into ω.

(Ia) There is a system as in (I) with the further property that, for each $\alpha < \omega_1$, any two functions f, g in F_α satisfy $f(\xi) = g(\xi)$ for all but finitely many $\xi < \alpha$.

(Ib) There is a system as in (I) such that if $\alpha < \omega_1$, and $f, g \in F_\alpha$, then $\mathrm{ran}(f)\Delta\mathrm{ran}(g)$ is finite, where Δ denotes symmetric difference.

(II) There exists a system of countable families of ladders, $\{\mathcal{L}_\alpha : \alpha < \omega_1,\ \alpha \text{ is a limit ordinal}\}$, such that each $L \in \mathcal{L}_\alpha$ is a ladder at α.

(IIa) There is a system as in (II) with the further property that, for each $\alpha < \omega_1$, any ladders L, M in $\mathcal{L}_\alpha$ satisfy $L(n) = M(n)$ for all but finitely many $n \in \omega$.

(IIb) There is a system as in (II) such that if $\alpha < \omega_1$, and $L, M \in \mathcal{L}_\alpha$, then $\mathrm{ran}(L)\Delta\mathrm{ran}(M)$ is finite.

Clearly (Ia) $\to$ (Ib) $\to$ (I) and (IIa) $\to$ (IIb) $\to$ (II). Also (I) $\to$ (II), (Ia) $\to$ (IIa), and (Ib) $\to$ (IIb); however, unlike in the case of the axioms in Theorem 1 that they mimic, the last three implications don't seem to reverse; neither do the others. One can however obtain conditions equivalent to (Ia), (Ib) and (I) respectively by asking that the members of each F_α be functions from ω onto α. Indeed, if $f : \alpha \to \omega$ is 1-1, then any left inverse of f is a function from ω onto α, while any right inverse of the latter kind of function is a function of the former kind. When the domain of a surjective function is well-ordered, ZF is already enough to produce a right inverse: take each element ξ in the range to the least element in $f^{\leftarrow}\{\xi\}$. This will be referred to as the "canonical right inverse" below.

(I) is an easy consequence of the following axiom, and also of its weakening ω-MC$(\aleph_1, 2^{\aleph_0})$, defined analogously to MC$(\aleph_1, 2^{\aleph_0})$:

ω-MC(∞, ∞): For every $\mathcal{A} = \{A_\gamma : \gamma \in \Gamma\}$ of non-empty pairwise disjoint sets there exists a family $\mathcal{F} = \{F_\gamma : \gamma \in \Gamma\}$ of countable non-empty sets such that for every $\gamma \in \Gamma$, $F_\gamma \subseteq A_\gamma$.

(IIb) is equivalent to the following statement: there is a locally compact topology on ω_1 refining the order topology, such that the set of limit ordinals forms a closed discrete subspace, but none are isolated.

If there is a special Aronszajn tree, then it is easy to show that (I) holds. As a partial converse, we have:

Theorem 6. *If* (I) *holds and* ω_1 *is singular, then there is a special Aronszajn tree.*

A model of (I) where ω_1 is singular is described below; thus "well-orderable" cannot be dropped from 3a. The following example witnesses the truth of Theorem 6, even if one adopts the restricted definition of "Aronszajn tree" in the cautionary note of Section 1. It is named after a fascinating botanical phenomenon, of which there is a fine example in my neighborhood.

Example 1. "The Witches'-Broom". This is a tree of height ω_1 whose existence uses a sequence $\beta_n \nearrow \omega_1$ of non-limit ordinals such that $\beta_0 = 0$ and a family $\{F_\alpha : \alpha < \omega_1\}$ as in (I). For each countable ordinal α, let n be the unique natural number such that $\beta_n \leq \alpha + 1 < \beta_{n+1}$, and let G_α be the set of all functions $g : \alpha + 1 \to (\omega \times \omega)$ for which there is a finite sequence $\sigma = \langle f_1, \dots f_{n+1}\rangle$ such that $f_i \in F_{\beta_i}$ and $g(\xi) = \langle i, f_i(\xi)\rangle$ for the unique i satisfying $\beta_i \leq \xi < \beta_{i+1}$. Let $T = \bigcup\{G_\alpha : \alpha < \omega_1\}$ ordered by end extension. Clearly, T is of height ω_1, and each level is countable because the members of each level are uniquely associated with finite sequences σ whose members are taken from countable sets. T cannot have an uncountable branch because that would give us a 1-1 function from ω_1 into $\omega \times \omega$; thus T is Aronszajn. T even satisfies the restrictive definition given in the cautionary note of Section 1. T is also special: for each $\langle n, k \rangle$, the following is an antichain: $\{g \in T : g(max(dom(g))) = \langle n, k \rangle\}$.

Because of Example 1 and Theorem 1, a negative answer to any part of the following would require large cardinals.
Problem 2. Is (I), or (Ib), or at least (Ia) enough to imply the existence of an Aronszajn tree? a special Aronszajn tree?

Theorem 7. *If* (Ib) *holds and ω_1 is singular, then there is an uncountable well-orderable set of reals.*

A slight modification of the Feferman-Lévy model produces a model of (Ia) and hence of (1b) in which ω_1 is singular ("Postulate C"). Where Feferman and Lévy use the group $\mathcal{G}$ of all permutations of $\omega \times \omega$ that leave the first coordinate invariant, this modification uses just the subgroup $\mathcal{H}$ of members of $\mathcal{G}$ which move only finitely many elements of $\omega \times \omega$. The definitions of the f_n are as before and all the arguments in Jech's *Set Theory,* pp. 213–4 go through without change.

In this modification, the set of all translates of any name for a set is countable. Thus if we start with the names $\dot{f}_n$ of each f_n and define

$$\dot{F}_n = \{\langle \pi \dot{f}_n, \mathbf{1} \rangle : \pi \in \mathcal{H}\}$$

then $\dot{F}_n$ is a symmetrical name of a countable set, and so is

$$\dot{\mathcal{F}}_n = \{\langle \dot{F}_n, \mathbf{1} \rangle : n \in \omega\}.$$

Each $\pi \dot{f}_n$ is the name of a function from ω onto the ω_n of the ground model $\mathfrak{M}$, a function that differs from f_n in only finitely many coordinates. By restricting each of these functions to each $\alpha \in \omega_n^{\mathfrak{M}}$, we get a system of countable families G_α of surjective functions from subsets of ω to α. From this system we can easily obtain a system satisfying (Ia) as explained earlier.

If the ground model $\mathfrak{M}$ satisfies MA + $\mathfrak{c} = \aleph_{\omega+1}$, then there is an uncountable set of sequences well-ordered by $<^*$ in the extension $\mathfrak{N}$. If $\mathfrak{M}$ satisfies GCH, there is no such set of sequences in $\mathfrak{N}$; however, by Theorem 7, there is an uncountable well-orderable set of reals in $\mathfrak{N}$ no matter what $\mathfrak{M}$ is. Thus, with the possible exception of (2) [see Problem 1], no statement in the following list implies the one before it—but, as we have seen, each one implies the later ones:

(1) There is a ladder system on ω_1.

(2) There is an (ω_1, ω_1^*)-gap.

(3) There is an uncountable set of functions from ω to ω that is $<^*$-well-ordered.

(4) There is an uncountable well-orderable subset of the real line.

Problem 3. If there is a system satisfying (Ia), is there a coherent one?

In other words, can we remove the restriction on f and g in (Ia) that requires them to belong to the same F_α, and ask that $f(\xi) = g(\xi)$ for all but finitely many ξ in the common domain of f and g? If ω_1 is singular and $\mathcal{F}$ satisfies (Ia), then the G_α in the construction of the Witches' Broom form a coherent system satisfying (Ia) with $\omega \times \omega$ substituted for ω. So a negative answer to Problem 3 implies the consistency of an inaccessible cardinal. This can also be seen from the fact, shown in the next section, that an affirmative answer to Problem 3 also implies one to the (Ia) part of Problem 2.

3. Aronszajn trees and long line smoothings from coherent families

Definition 3.1. Let $\langle A, \leq \rangle$ be a well-ordered set. An element $\alpha \in A$ is called a *limit in* A or *a limit element of* A if it has no immediate predecessor and is not the least element of A. An A-*sequence* is a function whose domain is A. We use $\langle r_\alpha : \alpha \in A \rangle$ to denote the A-sequence σ that satisfies $\sigma(\alpha) = r_\alpha$ for all $\alpha \in A$.

The following definition is typical of a genre of definitions by transfinite recursion. To define something in this genre for a well-ordered set A, we assume that it has already been defined for proper initial segments of A, and we break it up into the cases where A has a greatest element and where it does not. Such definitions are necessarily awkward because proper initial segments of A are also one of these two kinds, and we have to talk about one kind before we do of the other.

Definition 3.2. Let $\langle A, \leq \rangle$ be a countable well-ordered set, and let σ be an A-sequence of real numbers. If A is finite, with n elements, we let $\alpha(i)$ be the ith element of A and let $\sum \langle r_\alpha : \alpha \in A \rangle = \sum_{i=1}^{n} r_{\alpha(i)}$. If A is infinite and has a greatest element γ, we say $\sum \langle r_\alpha : \alpha \in A \rangle$ *converges to* r and write $\sum \langle r_\alpha : \alpha \in A \rangle = r$ just in case $\sum \langle r_\alpha : \alpha < \gamma \rangle$ converges to $r - r_\gamma$.

If A is infinite and has no greatest element, We say $\sum \langle r_\alpha : \alpha \in A \rangle = r$ iff (a) for every limit $\beta \in A$ there exists $r(\beta) \in \mathbb{R}$ such that $\sum \langle r_\alpha : \alpha < \beta \rangle = r(\beta)$ and (b) for each positive $\epsilon \in \mathbb{R}$, there exists $\delta \in A$ such that $|r - \sum \langle r_\xi : \xi < \eta \rangle| < \epsilon$ for all $\eta \geq \delta$.

Note the special case in which A is infinite and there is no limit element in A: in this case, A is order-isomorphic to the set of natural numbers, and we get a definition essentially like the usual one. Another special case worth noting is when A has exactly one limit element β and a greatest element γ. Then $\sum \langle r_\alpha : \alpha \in A \rangle$ exists iff $\sum \langle r_\alpha : \alpha < \beta \rangle$ "is a convergent series," as we say in analysis, and then

$$\sum \langle r_\alpha : \alpha \in A \rangle = \sum \langle r_\alpha : \alpha < \beta \rangle + \sum \langle r_\eta : \beta \leq r_\eta \leq \gamma \rangle.$$

and the rightmost sum is over a finite set, so we look to the second sentence of Definition 3.2 for its definition.

The following definition and theorem allow us to cut through the Gordian knot of Definition 3.2 in the special case where all terms of the A-sequence σ are positive.

Definition 3.3. If $\Sigma = \sum\langle r_\alpha : \alpha \in A\rangle$, we say Σ *is an absolutely convergent series* if $\sum\langle |r_\alpha| : \alpha \in A\rangle$ exists.

It is easy to show that absolute convergence of Σ implies convergence of Σ. The following generalizes the "advanced calculus" theorem that all rearrangements of an absolutely convergent series converge to the same real number.

Theorem 8. *If A is a countably infinite well-ordered set and Σ is absolutely convergent, and $f : \omega \to A$ is any bijection, then*

$$\sum\langle r_\alpha : \alpha \in A\rangle = \sum_{n=0}^{\infty} r_{f(n)}.$$

Corollary. *If A is a countably infinite well-ordered set and $\langle r_\alpha : \alpha \in A\rangle$ is a convergent series in which every term is positive, then for each $\epsilon > 0$ there are only finitely many α such that $r_\alpha \geq \epsilon$.*

We now adopt the following notation.

$$\mathbb{Q}^\Sigma = \{\langle q_\xi : \xi < \alpha\rangle \ : \ \alpha < \omega_1, \ \sum_\xi q_\xi \text{ converges, and } q_\xi \in \mathbb{Q}^+ \text{ for all } \xi < \alpha\}.$$

Lemma. *There is an uncountable coherent subtree of $\mathbb{Q}^\Sigma$ iff there is an uncountable coherent subset of*

$$W = \{\sigma : \alpha \to \omega \mid \sigma \textit{ is 1-1 and } \alpha \in \omega, \textit{ and } ran(\sigma) \textit{ is co-infinite in } \omega\}$$

It is easy to see that $\mathbb{Q}^\Sigma$ is $\mathbb{R}$-special and that every uncountable coherent subtree of $\mathbb{Q}^\Sigma$ is Aronszajn. These things are also true of W, which is a tree in the extension order: we can embed W in $\mathbb{Q}^\Sigma$ by sending each $\sigma \in W$ to the sequence with the same domain that takes $\xi \in \mathrm{dom}(\sigma)$ to $1/2^{\sigma(\xi)}$. This sends coherent subtrees of W to coherent subtrees of $\mathbb{Q}^\Sigma$, showing the reverse implication in the lemma. The forward implication is proven by listing $\mathbb{Q}^+$ in a sequence, $\{q_n : n \in \omega\}$, taking $\tau \in \mathbb{Q}^\Sigma$ to a 1-1 sequence τ' in $\mathbb{Q}^\Sigma$ such that always $\tau(\xi) \leq \tau'(\xi)$, and then taking τ' to the sequence σ with the same domain that takes $\xi \in \mathrm{dom}(\tau')$ to the unique n such that $\tau(\xi) = q_n$. Of course, σ has coinfinite range since at most finitely many $\tau'(\xi)$ are $> \epsilon$ for any $\epsilon > 0$. □

The following theorem, together with the modified Feferman-Lévy model of Section 2, establishes that the converse of Theorem 4 does not hold.

Theorem 9. *If there is an uncountable coherent subtree of* $\mathbb{Q}^\Sigma$*, then there is a smoothing of the long line.*

If we go all the way and assume the existence of a ladder system, then we can produce a special Aronszajn subtree of $\mathbb{Q}^\Sigma$ without having to knock out the limit levels as we did in the Witches'-Broom. Also the tree has the esthetic advantage of being Hausdorff (T_2) in the interval topology; this is equivalent to every chain that is bounded above having a least upper bound. From the Corollary above, it is an easy step to:

Lemma. *If* $\Sigma = \sum\langle r_\xi : \xi < \alpha\rangle$ *is a series of positive rational numbers that converges to a rational number, then for every* $q \in \mathbb{Q}^+$ *there is a series* Σ_q *that converges to* q *and differs from* Σ *in only finitely many summands.*

Theorem 10. *If there is a ladder system on* ω_1*, then there is a coherent Hausdorff special Aronszajn subtree* T *of* $\mathbb{Q}^\Sigma$ *in which every member converges to a rational number.*

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