D-spaces, trees, and an answer to a problem of Buzyakova

This paper was motivated by the following question, posed by Buzyakova [2, Question 3.6]:

Question 1. *If X is a space such that $e(Y) = l(Y)$ for all subspaces Y of X, is X a D-space?*

0.1. Definition. The *extent* of a space X, designated $e(X)$, is the supremum of the cardinalities of its closed discrete subspaces. The *Lindelöf degree (or: Lindelöf number)* of X, designated $l(X)$, is the least κ such that every open cover of X has a subcover of cardinality κ.

A well known elementary fact is that $e(X) \leq l(X)$ for all topological spaces X. But from now on, "space" will mean "Hausdorff space."

0.2. Definition. A *neighborhood assignment* or *neighbornet* on a space X is a family of sets indexed by the points of X, each one a neighborhood of the indexing point. A *D-space* [resp. *dually discrete space*] is a space X such that for every neighbornet $\mathcal{V} = \{V_x : x \in X\}$ there is a closed discrete subset [*resp.* a discrete subset] D of X such that $\{V_x : x \in D\}$ covers X.

In the above definition, we may confine our attention to those $\mathcal{V}$ whose members are open — the *open neighborhood assignments.* This is because if a shrinking of a neighbornet has a [closed] discrete subspace associated with it as above, then so does the original neighbornet. Hence one can also confine oneself to neighborhoods from a given base for the topology or a system of neighborhood bases for the points.

Question 1 is the seventh of ten open problems about D-spaces that were repeated by Eisworth in [E]. In Section 1 we give a ZFC counterexample that is a tree with the interval topology.

0.3. Definition. A *tree* is a poset (partially ordered set) in which the set of predecessors of each element is well-ordered.

If T is a tree and $t \in T$ then $t^{\downarrow} = \{x \in T : x \leq t\}$ and $\nabla_t(T)$ (also denoted ∇_t or $t^{\uparrow}$ if the tree is clear from context) is $\{x \in T : t \leq x\}$.

The *height* of $t \in T$, denoted $ht(t)$, is the order type of $t^{\downarrow}$, and if α is an ordinal then $T(\alpha) = \{t : ht(t) = \alpha\}$. Some authors write T_α for $T(\alpha)$. The *height of* T, denoted $ht(T)$, is the least α such that $T(\alpha) = \emptyset$.

0.4. Definition. If T is a tree, the *interval topology* (sometimes referred to simply as "the tree topology") on T is the topology whose base is the set of all intervals of the form $(s, t] = \{x \in T : s < x <\leq t\}$ together with all singletons m such that m is a minimal element of T.

In Sections 2 and 3 we address the general questions of when a tree is a D-space [respectively, dually discrete] in its interval topology. Despite the similarity in the definitions of the two concepts, the class of dually discrete spaces is radically larger than the class of D-spaces. But one thing the two classes share is the wide range of uncertainty as to which spaces do or do not belong. On the one hand, we do not know whether every subspace of a compact hereditarily Lindelöf space is dually discrete; on the other hand, we also do not know of even a consistent example of a normal, weakly θ-refinable space that is not a D-space. Even where the interval topology on trees is concerned, we have a big gap in our knowledge, exemplified by the following two contrasting problems:

Problem 1. *Is it a theorem of ZFC that every Aronszajn tree is dually discrete?*

Problem 2. *Is there a tree that is not dually discrete?*

Section 1. The main counterexample

1.1 Lemma. *A subset D of a tree T is closed discrete iff every infinite ascending sequence in D is unbounded above in T.*

1.2. Lemma. *A tree T has a cofinal closed discrete subspace iff it has a cofinal subset that is the countable union of antichains.*

1.3. Theorem. *If a tree is a D-space, then every branch is of countable cofinality, and the tree has a cofinal subset which is a closed discrete subspace.*

Proof. Let $V_t = t^\downarrow$ for all points $t \in T$. If D is as in the definition of a D-space, then D is obviously a cofinal closed discrete subspace of T.

If B is a branch of T, let $V_t = t^\downarrow$ for all $t \in B$, while if $t \notin B$ then, let $V_t = t^\downarrow$ if $t^\downarrow$ does not meet B. Otherwise, by our Hausdorff assumption, B is closed, so we can let V_t be any interval $(s, t]$ that does not meet B. Then if D is as before, $D \cap B$ must be countable and cofinal in B by Lemma 1. □

We will return to the general question of when a tree is a D-space in Section 2. For now, we just note that the converse of Theorem 1.3 is far from true: any tree can be embedded as a closed subtree of a tree with a cofinal antichain, and every closed subspace of a D-space is a D-space. But there are examples of trees in which every branch is countable, but which are not D-spaces. A Souslin tree is a consistent example: clearly, a Souslin tree does not have a countable cofinal subset; but every union of countably many antichains in a Souslin tree is countable. On the other hand, Souslin trees are dually discrete: see Theorem 2.9.

Here is a very different, ZFC example of a non-D-space which gives a negative answer to Question 1.

1.4. Example. Let E be a stationary, co-stationary subset of ω_1. Members of the tree $T(E)$ are the compact subsets of E, ordered by end extension $<_T$. That is, if c_1 and c_2 are compact subsets of E, then $c_1 <_T c_2$ iff $c_1 \subset c_2$ and $\alpha < \beta$ for all

$\alpha \in c_1$ and $b \in c_2 \setminus c_1$. Since E does not contain a club, every branch of $T(E)$ is countable.

1.5. Definition. Call a tree *robust* if for every $t \in T$ and every α such that $ht(t) < \alpha < ht(T)$, there exists $x \in T$ such that $t < x$ and $ht(x) = \alpha$. In other words, each point of T has successors at every level above its own.

For our next theorem, we use a definition of "Baire" that refers to the logicians' wedge topology (also known as the *Alexandroff discrete topology* [Ny]) on posets. This is the topology whose base is the set of all wedges $\nabla_t = \{x \in T : x \geq t\}$. This is not a T_2 topology, but it is the topology logicians refer to when they use the expressions "dense," "open," and "Baire" in the context of trees. These have simple order-theoretic characterizations: a dense set in this topology is one that is cofinal; an open set is one that is upwards-closed; and:

1.6. Definition. A poset is *ω-distributive* or *Baire* if every countable collection of cofinal, upwards-closed sets has cofinal intersection.

The following is well known folklore.

1.7. Theorem. *Let T be a robust tree of height ω_1 in which every chain is countable. The following are equivalent.*

1. *No subset of the form ∇_t has a cofinal subset which is the countable union of antichains.*
2. *T is Baire.*
3. *Forcing with T cannot collapse ω_1.*

1.8. Lemma. *$T(E)$ is robust.*

Proof. See [Ba, Lemma 3.7] or [F]. □

1.9. Theorem. *$T(E)$ is not a D-space.*

Proof. By 1.2, 1.3, 1.7 and 1.8, it is enough to show that $T(E)$ is Baire. This is shown in [T, Lemma 9.12] where $T(E)$ is called $U(E)$, except that the proof makes no mention of the essential ingredient that $T(E)$ is robust, implicitly used in getting extensions arbitrarily far up inside a countable elementary submodel. [Compare the proof of 2. below.] □

1.10 Theorem. *$T(E)$ has no Aronszajn subtrees.*

Proof. Let S be a subtree of $T(E)$ in which every level is countable. If $c \in S$ then the level of c in S is no greater than its level in T, which in turn is no greater than $max(c)$. Suppose S is uncountable; the following argument yields a contradiction.

For $\xi \in \omega_1$, let

$\alpha(\xi) = min\,\{\eta : max(c) \leq \eta$ for all c in the ξth level of $S\}$

Let $\alpha_0 = 0$,
$\alpha_{\nu+1} = \alpha(\alpha_\nu + 1)$;
if μ is a limit ordinal, let $\alpha_\mu = sup\{\alpha_\nu : \nu < \mu\}$.

Then $\{\alpha_\nu : \nu < \omega_1\}$ is a club, and so it meets the complement of E in a stationary set. If ν is a limit ordinal and c is on the α_νth level of S, then $\alpha_\nu \in c$; but if $\alpha_\nu \notin E$ this is impossible. □

1.11. Theorem. *Let T be a tree. Exactly one of the following is true.*

(1) *T either has an uncountable branch or a Souslin subtree.*
(2) *Every uncountable subset of T contains an antichain of the same cardinality.*

Proof. If (1) fails, let S be an uncountable subset of T.

If $|S| = \omega_1$ we use the fact that any uncountable tree without an uncountable branch is either a Suslin tree or it contains an uncountable antichain.

If $cf(|S|) > \omega_1$, then some level must meet S in a set of cardinality $|S|$, and this is a closed discrete subspace.

Finally, suppose $|S|$ is singular of cofinality ω or ω_1. If $cf(|S|) = \omega$, let $\{\kappa_n : n \in \omega\}$ be cofinal in $|S|$, with $\kappa_0 > \omega_1$, and let $S' = \{x \in S : |\nabla_x(S)| < |S|\}$.

Case 1: $|S'| = |S|$. In this case, if $|\nabla_x(S)| < \kappa_n$ for some n and all $x \in S'$, then the minimal members of S' are an antichain of size $|S|$. Otherwise, pick the least θ such that the θth level $S'(\theta)$ of S' is infinite, and pick distinct $t_n \in S'(\theta)$ such that $|\nabla_{t_n}(S)| > \kappa_n$. Since $\kappa_n > \omega_1$, there must be an antichain L_n in $\nabla_{t_n}(S)$ of cardinality $> \kappa_n$; then $\bigcup_{n=0}^{\infty} L_n$ is as desired.

Case 2. $|S'| < |S|$. Let $U = S \setminus S'$, let θ be the least ordinal such that $U(\theta)$ is infinite, and pick distinct t_n in $U(\theta)$ and antichains L_n as above.

If $cf(|S|) = \omega_1$, just replace countable sets with sets of size ω_1 in the above constructions. □

1.12. Corollary. *If X is any subspace of $T(E)$, then $e(X) = l(X)$. Moreover, extent is always attained (except for ω under some definitions of extent).*

Problem 3. Is $T(E)$ dually discrete?

Of course, if the answer to Problem 3 is negative, so is the one to Problem 2.

Section 2. When is a tree a D-space? When is it dually discrete?

Problem 4. Suppose T is a tree such that every closed subtree has a cofinal subset that is the countable union of antichains. **(a)** Is T dually discrete in its interval topology? **(b)** Is it a D-space? **(c)** Is it quasi-metrizable?

2.2 Definition. A *quasi-metric* on a set X is a function $d : X^2 \to \mathbb{R}^+ \cup \{0\}$ such that:

(1) $d(x, y) = 0$ if, and only if, $x = y$.
(2) $d(x, z) \leq d(x, y) + d(y, z)$ for all x, y, z in X.

A space X is *quasi-metrizable* if there is a quasi-metric d on X such that the collection $(\{B_\epsilon(x) : x \in X, \epsilon > 0\}$ of open ϵ-balls is a base for the topology on X.

The converse of Problem 4(c) has an affirmative answer:

2.3. Theorem. *Every quasi-metrizable tree has a cofinal subset that is the countable union of antichains.*

An immediate corollary is that $T(E)$ is not quasi-metrizable.

Before proving Theorem 2.3, we introduce some concepts which will be useful later on.

2.4. Notation. Given a tree T of height $\leq \omega_1$, let
$R(T) = \{x \in T : \nabla_x \text{ is of height } \omega_1\}$.
Let $R_0(T) = T$. With $R_\alpha(T)$ defined, let $R_{\alpha+1}(T) = R(R_\alpha(T))$ and if α is a limit ordinal, let $R_\alpha(T) = \bigcap\{R_\beta(T) : \beta < \alpha\}$. Let $RC(T) = R_\alpha(T)$ for the least α such that $R_{\alpha+1}(T) = R_\alpha(T)$.

It is easy to see that if $RC(T)$ is nonempty, then it is robust; and in this case we refer to $RC(T)$ as the *robust core of* T.

Note how the following proof only uses the fact that $ht(T) \leq \omega_1$ until the Claim.

Proof of Theorem 2.3. Let T be a quasi-metrizable tree. If $RC(T) = \emptyset$ then every point of T is $\leq$ some member of $T \setminus R(T)$, which in turn has a cofinal subset that is a countable union of antichains: let A be the set of minimal members of $T \setminus R(T)$, and for each $x \in A$ let the levels of ∇_x be listed as $\{A_n(x) : n \in \omega\}$; let $A_n = \bigcup\{A_n(x) : x \in A\}$. Clearly, each A_n an antichain and $C = \bigcup\{A_n : n \in \omega\}$ is cofinal in T.

So suppose $RC(T) \neq \emptyset$. Every point of $T \setminus RC(T)$ is below some point of A. So it is enough to show that $RC(T)$ has a cofinal subset that is the countable union of antichains. This will follow from Theorem 1.7 and:

Claim. If T is quasi-metrizable and $RC(T) \neq \emptyset$, then forcing with $RC(T)$ collapses ω_1.

Proof of Claim. As seen from the proof of 1.7, forcing with a robust tree of height ω_1 adds a branch that meets every level, so that if ω_1 is not collapsed, the branch is a copy of ω_1, hence countably compact. However, quasi-metrics cannot be destroyed by forcing, and every countably compact, quasi-metrizable space is metrizable. This is because every quasi-metrizable space is a γ-space [G, 10.2, 10.5], and every countably compact g-space is metrizable [G, 10.8]. But ω_1 is not metrizable. $\square$

From the proof of the claim it can be deduced that every chain in a quasi-metrizable tree is countable.

2.5. Definition. Let L be a totally ordered set. A tree T is *L-special* if there is a $<$-preserving function from T to L. A tree is *special* if it is a countable union of antichains.

Problem 5. Is a tree a D-space (or: dually discrete) if it is **(a)** quasi-metrizable? **(b)** $\mathbb{R}$-special?

An affirmative answer to **(a)** also implies one to **(b)**: if T is $\mathbb{R}$-special, witnessed by $f : T \to \mathbb{R}$, then the following is a quasi-metric that generates the topology: $d(t, x) = min\{1, f(x) - f(t)\}$ if $t \leq x$, $d(t, x) = 1$ otherwise.

If a tree is hereditarily a D-space, then it cannot contain a copy of ω_1 and so every chain is countable. This motivates:

Problem 6. If a tree is a D-space with no uncountable chains, is it hereditarily a D-space if it is **(a)** quasi-metrizable? **(b)** $\mathbb{R}$-special?

Problem 7. If a tree is hereditarily a D-space, is it **(a)** quasi-metrizable? **(b)** $\mathbb{R}$-special? **(c)** special?

Now we turn to positive results. First, we show the converse of the last part of Problem 7.

2.6. Theorem. *Every special tree is hereditarily a D-space.*

Proof. A routine induction shows that if X is the countable union of closed D-subspaces, then X is a D-space. The rest is immediate from the fact that an antichain in a tree is closed discrete, and hence a D-space. $\square$

A similar proof shows that every locally compact, subparacompact space is a D-space. Simply use the following definition of "subparacompact": every open cover has a σ-discrete closed refinement; and, of course, the fact that every compact space is a D-space.

Next, we show that the extra conditions (**(a)**, etc.) in Problem 7 are important.

2.7. Definition. The *branch completion* of a tree T is the tree $\tilde{T}$ obtained by adjoining a point t_B at the end of each branch B of T. That is, $\tilde{T} = T \cup \{t_B :$ B is a branch of $T\}$ and if $t_1, t_2 \in \tilde{T}$ then $t_1 \leq_{\tilde{T}} t_2$ iff either $t_i \in T$ for $i = 1, 2$ and $t_1 \leq_T t_2$ or $t_1 \in T$ and $t_2 = t_B$ for some branch B such that $t_1 \in B$.

For simplicity, we write $\tilde{T}(E)$ for $\widetilde{T(E)}$ below. It is a tree in which every chain is countable, and $T(E)$ is a dense, downwards closed (hence open) subtree which is not a D-space. However, $\tilde{T}(E)$ is a D-space (accounting for the extra conditions in Problem 5):

2.8 Theorem. *$\tilde{T}(E)$ is a D-space.*

Proof. Let $\mathcal{U} = \langle U_t : t \in \tilde{T}(E) \rangle$ be an open neighbornet and let $D_0 = \tilde{T}(E) \setminus T(E)$. Let $U = \bigcup \{U_d : d \in D_0\}$ and let $S = \tilde{T}(E) \setminus U = T(E) \setminus U$.

Claim. S is special.

Once the claim is proved, use Theorem 2.6 and the fact that S is closed in $\tilde{T}(E)$ (and hence the relative topology on S is the interval topology) to get a closed discrete subspace D_1 of $\tilde{T}(E)$ such that $S \subset \bigcup \{U_d : d \in D_1\}$. Then $\tilde{T}(E) = \bigcup \{U_d : d \in D_0 \cup D_1\}$, as desired.

Proof of Claim Otherwise, let $S' = S \setminus \{s \in S : \nabla_s(S) \text{ is special }\}$. Then $S \setminus S'$ is special, and we will be done once we show that $S' = \emptyset$.

Suppose $S' \neq \emptyset$. Then $ht(S') = \omega_1$ and S' is robust. Let N_0 be a countable elementary submodel of a sufficiently large fragment of the universe containing $E, T(E)$, and $\mathcal{U}$. Then $\tilde{T}(E)$, D_0, and U are also elements of N_0. Let $\{N_\alpha : \alpha < \omega_1\}$ be a continuous $\in$-chain of countable elementary submodels. Let $\delta \notin E$ be such that $N_\delta \cap \omega_1 = \delta$, and let $\alpha_n \nearrow \delta$. Note that $\alpha_n \in N_\delta$ for each n.

Let $x_0 \in S' \cap N_\delta$. By elementarity, there exists $x_1 \in S' \cap N_\delta$ such that $x_1 \geq x_0$ and the height of x_1 in S' is at least α_1 (but $< \delta$). In general, with x_n defined, let $x_{n+1} \geq x_n$, $x_{n+1} \in S' \cap N_\delta$, $\alpha_{n+1} \leq ht_{S'}(x_{n+1}(< \delta)$. Then the set of all x_n is not bounded above in $T(E)$ and hence it determines a branch B of $T(E)$. But then t_B is in the closure of $\{x_n : n \in \omega\}$, a contradiction. $\square$

Here is a theorem promised in Section 1.

2.9. Theorem. *Every Souslin tree is dually discrete.*

Proof. If T is an Aronszajn tree and N is a neighbornet, call a point *popular* if it is in uncountably many sets $N(t)$, otherwise call it *unpopular*. A standard argument shows that the following is a club subset of ω_1:

$$C = \{\alpha : ht(t) < \alpha \text{ and } t \text{ is unpopular implies } ht(x) < \alpha \text{ for all } x \text{ such that } t \in N(x)\}$$

This club divides up the unpopular points into countable relatively clopen sets, and the unpopular points on levels indexed by C form a closed discrete subspace.

However, if T is Souslin, then every closed discrete subspace is countable, so there are only countably many unpopular points altogether.

Fix δ so that all unpopular points are below level δ. The levels at δ or below form a countable clopen D-subspace T_δ. Let D_0 be a closed discrete subspace of T_δ and hence of T such that $\{N(t) : t \in D_0\}$ covers T_δ.

List the points above level δ as $\{p_\alpha : \alpha \in \omega_1\}$ and inductively choose x_α so that $p_\alpha \in N(x_\alpha)$, the level of x_α is above that of the earlier x_β, and so that whenever α is a limit ordinal, then there is a limit ordinal between the levels of the x_β, $\beta < \alpha$ and the level of x_α itself. The resulting set D_1 of x_α is discrete, and so is $D_0 \cup D_1$, and $\{N(t) : t \in D_1 \cup D_2\}$ covers T. $\square$

The foregoing argument can be easily adapted to *almost Souslin trees*, which are Aronszajn trees in which every antichain meets a nonstationary set of levels. But there is, consistently, a big no-man's land of Aronszajn trees between the almost Souslin and the special, where the foregoing argument runs into major difficulties.

3. Additional Properties of $T(E)$

We have already seen that $T(E)$ is not quasi-metrizable. It also fails to satisfy the nice topological properties that Eric van Douwen was interested in, and which are featured in most of the other problems in [E].

A nice property that $T(E)$ does have is realcompactness. This property is characterized by every Z-ultrafilter with the countable intersection property being fixed:

Definition. A *zero-set* of a space X is a set of the form $f^{\leftarrow}\{0\}$ for some continuous function $f : X \to \mathbb{R}$. A *Z-ultrafilter* on a space X is an ultrafilter $\mathcal{U}$ of the lattice of zero-sets. That is, if Z_0 and Z_1 are zero-sets in $\mathcal{U}$, then their intersection is in $\mathcal{U}$, every zero-set containing one in $\mathcal{U}$ is itself in $\mathcal{U}$, the empty set is not in $\mathcal{U}$, and if Z is a zero-set not in $\mathcal{U}$ then it is disjoint from some member of $\mathcal{U}$.

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