A CORSON COMPACT L-SPACE FROM A SOUSLIN TREE

Peter Nyikos

Abstract. The completion of a Souslin tree is shown to be a consistent example of a Corson compact L-space when endowed with the coarse wedge topology. The example has the further properties of being zero-dimensional and monotonically normal.

1. Introduction

In this paper, the coarse wedge topology on trees is used to construct what may be the first consistent example of a Corson compact L-space that is monotonically normal. It is considerably simpler and easier to (roughly!) visualize than the CH example of a Corson compact L-space produced by Kunen [4] or the Corson compact L-space produced by Kunen and van Mill under the hypothesis that 2^{ω_1} with the product measure is the union of a family $\aleph_1$ nullsets, such that every nullset is contained in some member of the family [5].

Corson compact L-spaces cannot be constructed in ZFC alone, because MA_{ω_1} implies there are no compact L-spaces at all. This is one of the earliest applications of MA_{ω_1} to set-theoretic topology, and one of the few that uses its topological characterization, *viz.,* that a compact ccc space cannot be the union of $\aleph_1$ nowhere dense sets [3], [9, 6.2], [10, p. 16].

Recall that a *Corson compact space* is a compact Hausdorff space that can be embedded in a Σ-product of real lines, *viz.,* the subspace of a product space $\mathbb{R}^\Gamma$ (for some set Γ) consisting of all points which differ from the zero element in only countably many coordinates. Corson compact spaces play a role in functional analysis, especially through their spaces of continuous functions, the Banach space $\langle C(K), \|\cdot\|_\infty\rangle$ and $C_p(K)$, the space of real-valued continuous functions with the relative product topology.

1991 *Mathematics Subject Classification.* (Updated to 2013.) Primary: 46A40, 54A25, 54D35, 54D80 Secondary: 54A35, 54B99.

Key words and phrases. Corson compact, Σ-product, L-space, tree, level, chain complete, Dedekind complete, chain completion, coarse wedge topology, Souslin tree, monotonically normal
Email address: nyikos@math.sc.edu.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Recall that a topological space is *separable* iff it has a countable dense subset, and *Lindelöf* iff every open cover has a countable subcover. The following terminology is now standard:

Definition 1.1. *An L-space is a regular, hereditarily Lindelöf space which has a non-separable subspace.*

For about four decades, one of the best known unsolved problems of set-theoretic topology was whether there is a ZFC example of an L-space. This was solved in an unexpected manner by Justin Tatch Moore, who constructed one with the help of a deep analysis of walks on ordinals [6]. The following problem, motivated by our main example, may still be unsolved:

Problem 1.2. *Is there a ZFC example of an L-space which embeds in a Σ-product of real lines?*

Definition 1.3. A space X is *monotonically normal* if there is a function $U(E,F)$ defined on pairs of disjoint closed sets $\langle E,F\rangle$ such that: (1) $U(E,F)$ is an open set; (2) $E \subset U(E,F)$ and $U(E,F) \cap U(F,E) = \emptyset$; and (3) if $E \subset E'$ and $F \supset F'$, then $U(E,F) \subset U(E',F')$.

A neat feature of our main example is that it is being monotonically normal, and is thus the continuous image of a compact orderable space [11] — and yet every linearly orderable Corson compact space is metrizable [1]. One natural question is whether the main example is actually the continuous image of a compact orderable L-space: such spaces exist iff there is a Souslin tree/line. A much more general pair of contrasting questions may be open:

Problem 1.4. *Is the existence of a monotonically normal compact L-space equivalent to the existence of a Souslin tree?*

Problem 1.5. *Is there a ZFC example of a monotonically normal L-space?*

2. Trees and the coarse wedge topology

The purpose of this section is to make this paper as self-contained as reasonable, and to show that trees with the coarse wedge topology have a property even stronger than being monotonically normal. Readers with a good understanding of trees might try omitting it on a first reading.

Definition 2.1. A *tree* is a partially ordered set in which the predecessors of any element are well-ordered. [Given two elements $x < y$ of a poset, we say x is a *predecessor* of y and y is a *successor* of x.]

Definition 2.2. If a tree has only one minimal member, it is said to be *rooted* and the minimal member is called the *root* of the tree. A *chain* in a poset is a totally ordered subset. An *antichain* in a tree is a set of pairwise incomparable elements. Maximal members (if any) of a tree are called *leaves*, and maximal chains are called *branches*.

Definition 2.3. If T is a tree, then $T(0)$ is its set of minimal members. Given an ordinal α, if $T(\beta)$ has been defined for all $\beta < \alpha$, then $T \restriction \alpha = \bigcup\{T(\beta) : \beta < \alpha\}$, while $T(\alpha)$ is the set of minimal members of $T \setminus T \restriction \alpha$. The set $T(\alpha)$ is called *the α-th level of T*. The *height* or *level* of $t \in T$ is the unique α for which $t \in T(\alpha)$, and it is denoted $\ell(t)$. The *height of T* is the least α such that $T(\alpha) = \emptyset$.

The following example illustrates some fine points of associating ordinals with trees and their elements.

Example 2.4. The *full ω-ary tree of height $\omega + 1$* is the set T of all sequences of nonnegative integers that are either finite or have domain ω, and in which the order is end extension. Each chain of order type ω consists of finite sequences whose union is an ω-sequence on level ω. Since this is the last nonempty level of the tree, the tree itself is of height $\omega + 1$. The subtree $T \restriction \omega$ is the *full ω-ary tree of height ω*.

Definition 2.5. A tree is *chain-complete* [resp. *Dedekind complete*] if every chain [*resp.* chain that is bounded above] has a least upper bound. A tree is *complete* if it is rooted and chain-complete.

Definition 2.6. For each t in a tree T we let V_t denote the wedge $\{s \in T : t \leq s\}$. The *coarse wedge topology* on a tree T is the one whose subbase is the set of all wedges V_t and their complements, where t is either miminal or on a successor level.

Because of the way trees are structured, the nonempty finite intersections of members of the subbase are "notched wedges" of the form

$$W_t^F = V_t \setminus \bigcup\{V_s : s \in F\} = V_t \setminus V_F$$

where F is a finite set of successors of t.

If t is minimal or on a successor level, then a local base at t is formed by the sets W_t^F such that F is a finite set of immediate successors of t. If, on the other hand, t is on a limit level, then a local base is formed by the W_s^F such that s is on a successor level below t.

It is easy to see that a tree is Hausdorff in the coarse wedge topology iff it is Dedekind complete. In particular, if C is a chain that is bounded above but has no supremum, then it converges to more than one point.

A corollary of the following theorem is that every complete tree is compact Hausdorff in the coarse wedge topology.

Theorem 2.7. [7, Corollary 3.5] *A tree is compact Hausdorff in the coarse wedge topology iff it is chain-complete and has only finitely many minimal elements.*

Theorem 2.8. *A complete tree is Corson compact in the coarse wedge topology iff every chain is countable.*

Proof. A necessary and sufficient condition for a compact space being Corson compact is that it have a point-countable T_0 separating cover by cozero sets—equivalently, open F_σ-sets [1]. If the complete tree has an uncountable chain, then it has a copy of $\omega_1 + 1$, which does not have a point-countable T_0-separating open cover of any kind, thanks in part to the Pressing-Down Lemma (Fodor's Lemma).

Conversely, if every chain is countable, then the clopen sets of the form V_t clearly form a T_0-separating, point-countable cover, and T is compact Hausdorff by Theorem 2.7. □

Hausdorff trees with the coarse wedge topology have a property even stronger than monotone normality; it is the property that results if "clopen" is substituted for "open" in Definition 1.3:

Definition 2.9. A space X is *monotonically ultranormal* if there is a function $U(E, F)$ defined on pairs of disjoint closed sets $\langle E, F\rangle$ such that: (1) $U(E, F)$ is a clopen set; (2) $E \subset U(E, F)$ and $U(E, F) \cap U(F, E) = \emptyset$; and (3) if $E \subset E'$ and $F \supset F'$, then $U(E, F) \subset U(E', F')$.

The property in the following theorem is named with the Borges criterion [see below] for monotone normality in mind.

Theorem 2.10. [8, Theorem 2.2] *Every Hausdorff space satisfying the following property is monotonically ultranormal.*

Property B+. To each pair $\langle G, x\rangle$ where G is an open set and $x \in G$, it is possible to assign an open set G_x such that $x \in G_x \subset G$ so that $G_x \cap H_y \neq \emptyset$ implies either $x \in H_y$ or $y \in G_x$.

The Borges criterion puts H for H_y and G for G_x in the part of Property $B+$ after "implies."

The question of whether every monotonically ultranormal Hausdorff space satisfies Property B+ was posed in [8] and is still open.

Theorem 2.11. *Every Hausdorff tree with the coarse wedge topology has Property B+.*

Proof. For each point t and each open neighborhood G of t, there exists $s \le t$ for which there is a basic clopen set W_s^F such that $t \subset W_s^F \subset G$, and for which $F \subset V_t$. [If t is on a successor level we can let $s = t$, while if t is on a limit level we first find some $s' < t$ on a successor level and finite $F' \subset V_s$ for which $t \subset W_{s'}^{F'}$; then let $F = F' \cap V_t$ and, using Dedekind completeness, choose s such that $s' \leq s < t$ and all elements of $F' \setminus F$ are incomparable with s. Then W_s^F is as desired.]

Now for each $x \in F$ let x' be the immediate successor of t below x and let $F^* = \{x' : x \in F\}$.

Claim. *Letting $G_t = W_s^{F^*}$ for each t, G as above produces an assignment witnessing Property B+.*

Proof of Claim. The notched wedges W_t^F clearly have the property that the intersection of any two contains the minimum point of one of them. Let $G_x \cap H_y \neq \emptyset$. Assume that the minimum point t of G_x is in H_y; in particular, $t \geq s$. Let $H_y = W_s^{F^*}$.

Case 1. $y < t$. Then $G_x \subset V_t \subset H_y$, because t is not in $V_{z'}$ for any $z' \in F^*$.

Case 2. y and t are incomparable. Then $t > s$, and we again have $G_x \subset V_t \subset H_y$.

Case 3. $t \leq y$. Then if x and y are incomparable, we clearly have $s < x \in H_y$. This also holds if $x \leq y$. Finally, if $x > y$, we must have $y \in G_x$. □

Corollary 2.12. *Every Hausdorff tree is monotonically normal in the coarse wedge topology.*

3. The main example

The following construction is utilized in the main example of this paper.

Example 3.1. For any tree T, we call a tree *a completion* of T if it is formed by adding a supremum to each downwards closed chain that does not already have one. Formally, we define *the completion* $\hat{T}$ *of* T as follows. If T is not rooted, we let $\hat{T}$ be the collection of downwards closed chains (called "paths" by Todorčević), ordered by inclusion. If T is rooted, we only put the nonempty paths in $\hat{T}$.

We identify each $t \in T$ with the path $P_t = \{s \in T : s \leq t\}$. Completeness of $\hat{T}$ follows from rootedness of $\hat{T}$ and from the easy fact that the supremum of a chain C of $\hat{T}$ is the same as the supremum of $C \cap T$. In particular, if C is a path in $\hat{T}$ then $C \cap T$ is downwards closed in T.

Todorčević called the set of characteristic functions of the paths of T *the path space of* T when endowed with the topology inherited from the product topology on 2^T. Gary Gruenhage [2] showed that this topology is the coarse wedge topology of $\hat{T}$.

Recall that a *Souslin tree* is an uncountable tree in which every chain and antichain is countable. Let us call a tree *uniformly ω-ary* if every nonmaximal point has denumerably many immediate successors. [For instance, Example 2.4 is a uniformly ω-ary tree.]

As is well known, every Souslin tree has a subtree T in which every point has more than one successor at every level above it. Thus every point of T has denumerably many successors on the next limit level above it. And so, a uniformly ω-ary Souslin tree results when we take the subtree S of all points on limit levels of T.

Theorem 3.2. *The completion $\hat{S}$ of a uniformly ω-ary Souslin tree S is an L-space in the coarse wedge topology.*

Proof. Since $\hat{S} \restriction \alpha + 1$ is closed for all $\alpha < \omega_1$, $\hat{S}$ is not separable. In the proof that $\hat{S}$ is hereditarily Lindelöf, uniform ω-arity plays a key role: if the tree were finitary, every point on a successor level would be isolated.

We make use of the elementary fact that a space is hereditarily Lindelöf if (and only if) every open subspace is Lindelöf. Let W be an open subspace of $\hat{S}$, and let W_0 be the set of points $t \in W$ such that $V_t \subset W$. If $t \in W_0$ is on a limit level, there is also $s < t$ such that V_s is clopen and $s \in W_0$: see the first paragraph in the proof of Theorem 2.11, and note that here, $F = \emptyset$. Let $A = \{a \in W_0 : a \text{ is minimal in } W_0\}$. Then W_0 is the disjoint union of the clopen wedges V_α $(a \in A)$, and A is countable by the Souslin property.

If $x \in W \setminus W_0$, then there is a basic clopen subset of W of the form W_t^F where $F \neq \emptyset$ and $F \subset V_x$: see the first paragraph in the proof of 2.11 again. There are no more than $|F|$ immediate successors of x below some element of F, and if s is one of the other immediate successors of x, then $V_s \subset V_x \setminus V_F$, so $s \in W_0$. But then $s \in A$ also, since any V_z containing V_s properly must also contain x, contradicting $x \in W \setminus W_0$. So $W \setminus W_0$ is countable, and we have countably many basic clopen sets whose union is W. □

The following is now immediate from 2.8, 2.12, and 3.3.

Corollary 3.3. *If there is a Souslin tree, there is a Corson compact, monotonically normal L-space.*

References

[1] V. A. Efimov and G.I. Čertanov, "Subspaces of Σ-products of intervals." (Russian. English summary) Comment. Math. Univ. Carolin. 19 (1978), no. 3, 569–593.

[2] G. Gruenhage, "On a Corson compact space of Todorčević," Fund. Math. 126 (1986), no. 3, 261–268.

[3] I. Juhász, "Martin's axiom solves Ponomarev's problem," Bull. Acad. Polon. Sci. Sér. Sci. Math. Astron. Phys. 18 (1970) 71–74.

[4] K. Kunen, "A compact L-space under CH," Top. Appl. 12 (1981) no. 3, 283–287.

[5] K. Kunen and J. van Mill, "Measures on Corson compact spaces," Fund. Math. 147 (1995) 61–72.

[6] J. T. Moore, "A solution to the L space problem," J. Amer. Math. Soc. 19 (2006), no. 3, 717–736.

[7] P. Nyikos, "Various topologies on trees," *Proceedings of the Tennessee Topology Conference*, P.R. Misra and M. Rajagopalan, eds., World Scientific Publishing Co., 1997, pp. 167–198.
Electronic ps and pdf copies are linked at http://arxiv.org/abs/math/0412554

[8] P. Nyikos, "Metrizability, monotone normality, and other strong properties in trees," Top. Appl. 98 (1999) 269–290.

[9] J. Roitman, "Basic S and L," *Handbook of Set-Theoretic Topology,* K. Kunen and J. Vaughan ed., North-Holland (1984) 295–326.

[10] M. E. Rudin, *Lectures on Set Theoretic Topology,* Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, no. 23, Amer. Math. Soc., Providence, R.I., 1975.

[11] M.E. Rudin, "Nikiel's conjecture," Topology Appl. 116 (2001), no. 3, 305–331.

Department of Mathematics, University of South Carolina, Columbia SC 29208