# SEQUENTIAL COMPACTNESS VS. COUNTABLE COMPACTNESS

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Abstract. The general question of when a countably compact topological space is sequentially compact, or has a nontrivial convergent sequence, is studied from the viewpoint of basic cardinal invariants and small uncountable cardinals. It is shown that the small uncountable cardinal $\mathfrak{h}$ is both the least cardinality and the least net weight of a countably compact space that is not sequentially compact, and that it is also the least hereditary Lindelöf degree in most published models. Similar results, some definitive, are given for many other cardinal invariants. Special attention is paid to compact spaces. It is also shown that MA($\omega_1$) for $\sigma$-centered posets is equivalent to every countably compact $T_1$ space with an $\omega$-in-countable base being second countable, and also to every compact $T_1$ space with such a base being sequential. No separation axioms are assumed unless explicitly stated.

## 1. Introduction

This article continues the theme, begun in [Ny], of sequential compactness (and lack thereof) in countably compact topological spaces, without the usual assumption of separation axioms. We do mention (and, in a few places, prove) some results involving the separation axioms $T_1$, KC, Hausdorff ($T_2$) and $T_3$ (= $T_2$ and regular) but we will always spell these axioms out when they are assumed.

In [Ny] one of us gave some reasons for taking this unusual (for him) step. These will not be repeated here, but there is an additional reason, which is behind practically all the results in this paper: quite unexpectedly, we have found countably compact spaces to be quite amenable to the techniques of modern set theory even in a general topological setting. The "small uncountable cardinal" $\mathfrak{h}$ in particular plays a major role, as do the splitting trees which give one way of defining it.

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We need to choose our definition of "countably compact" carefully. Saying "every infinite subset has an accumulation point," as in the usual statement of the Bolzano-Weierstrass theorem, gives rise to such unintended examples as the product of $\omega$ with the indiscrete 2-point space. Engelking's text [E] even goes so far as to restrict "countably compact" to Hausdorff spaces. However, the alternative definition, "every countable open cover has a finite subcover," turns out to give a nice structure theory even for general spaces. A standard exercise is that it is equivalent to "every sequence has a cluster point" and also to "every denumerable subset has a complete accumulation point" [that is, a point $x$ such that every neighborhood of $x$ contains infinitely many elements of the denumerable subset]. As usual, "sequentially compact" is the strictly stronger condition that every sequence has a convergent subsequence.

For most of this paper, the general theme will be how "small" a countably compact or compact space can be with respect to the basic cardinal invariants [J] [Ho2] without being sequentially compact. In our next to last section we deal with a more specialized concept, that of a $\kappa$-in-$\lambda$ and a $\kappa$-in-$<\lambda$-base, because our techniques in the earlier sections adapt readily to this context too.

In the final section we also study the related general theme of how "small" an infinite [countably] compact space can be and have only trivial convergent sequences, and also touch on this theme in Section 3.

Here is a piquant twist on the usual assumptions on separation axioms: *We may as well confine our attention to $T_0$ spaces.* This is because all the basic cardinal invariants on spaces are unaffected by the passage from a space $X$ to its $T_0$ reflection (except for cardinality, which may be reduced, but that is in line with our general themes), as are compactness, countable compactness, sequential compactness, sequentiality, the base properties in the next to last section, as are the negations of these properties. All this is obvious from the following description of the $T_0$ reflection of a space $X$: it is the quotient space of $X$ modulo the equivalence relation, $x \equiv y \iff$ every open set containing $x$ contains $y$, and vice versa. This induces an isomorphism of the topologies of $X$ and $X/\equiv$, from which all else follows.

Where the second general theme is concerned, we can confine ourselves to the class of $T_1$ spaces, because if every open set containing $x$ also contains $y$, then any sequence which alternates between $x$ and $y$ will converge to $x$.

With one possible exception (the least number of open sets) we can even confine ourselves to the class of $T_1$ spaces where the first general theme is concerned. The reasons for this are varied and will be made clear in the last section. One of the tools we use there is the preorder $x \leq y$ iff $x \in c\ell\{y\}$ iff every open set containing $x$ contains $y$. This is a true partial order on $T_0$ spaces. In more general spaces, the conjunction of $x \leq y$ and $y \leq x$ is equivalent to $x \equiv y$.

The small uncountable cardinals $\mathfrak{h}$, $\mathfrak{s}$, and $\mathfrak{t}$ play a major role below. The cardinal $\mathfrak{t}$ is the least cardinality of a complete tower on $\omega$. By a *complete tower* we mean a collection of sets well-ordered wrt reverse almost containment ($A \leq B$ iff $B \setminus A$ is finite, written $A \subseteq^* B$) such that no infinite set is almost contained in every member of the collection. The cardinals $\mathfrak{s}$ and $\mathfrak{h}$ are defined with the help of the following concepts. A set $S$ is said to **split** a set $A$ if both $A \cap S$ and $A \setminus S$ are infinite. A *splitting family on* $\omega$ is a family of subsets of $\omega$ such that every infinite subset of $\omega$ is split by some member of the family. We will call a splitting family *a splitting tree* if any two members are either almost disjoint, or one is almost contained in the other; thus it is a tree by reverse almost inclusion.

The least cardinality of a splitting family is denoted $\mathfrak{s}$, while least height of a splitting tree is denoted $\mathfrak{h}$. It is easy to show that $\omega_1 \leq \mathfrak{t} \leq \mathfrak{h} \leq \mathfrak{s} \leq \mathfrak{c}\,(= 2^\omega)$. For more about the relationships of these cardinals see [D1] and [V] and (except for $\mathfrak{h}$) [vD], and also some of the consistency results mentioned below. The seminal paper on $\mathfrak{h}$ is [BPS], where it is denoted $\varkappa(N^*)$. The usual correspondence between subsets of $\omega$ and clopen subsets of $\beta\omega - \omega = \omega^*$ gives rise to a correspondence between splitting trees and what are called shattering refining matrices in [BPS].

Proofs of the following may be found in Section 6 of [vD]:

**Theorem A.** *$\mathfrak{c}$ is the least cardinality of a countably compact $T_3$ space that is not sequentially compact.*

**Theorem B.** *Every compact Hausdorff space of cardinality $< 2^\mathfrak{t}$ is sequentially compact.*

**Theorem C.** *$\mathfrak{s}$ is the least weight of a countably compact space [also of a compact space] that is not sequentially compact.*

The proof of Theorem C in [vD, Theorem 6.1] works for all countably compact spaces, even though [vD] uses the convention that “space” means “$T_3$ space.”

Obviously, Theorem A is definitive for countably compact $T_3$ spaces, and it is well known to extend to Urysohn spaces (*i.e.,* spaces in which distinct points have disjoint closed neighborhoods) but it already fails for Hausdorff spaces (see comments following the proof of Theorem 3 below). Theorem B is not optimal in all models of set theory: in Section 4, we show (Example 6) that there is a model in which every compact space of cardinality $2^\mathfrak{t}$ is sequentially compact, negatively answering the following question of van Douwen [vD, Question 6.6, in effect]:

**Question.** Is ZFC enough to imply the existence of a compact Hausdorff space of cardinality $2^\mathfrak{t}$ that is not sequentially compact?

It is not known whether Theorem B extends to compact spaces in general:

**Problem 1.** Is every compact space of cardinality $< 2^\mathfrak{t}$ sequentially compact?

Alas and Wilson [AW] have extended Theorem B to a wider class of spaces:

**Theorem B+.** *Every compact KC space of cardinality $< 2^{\mathfrak{t}}$ is sequentially compact.*

A **KC space** is one in which every compact subset is closed. An elementary theorem of general topology is that every Hausdorff space is KC. Even more elementary is the fact that every KC space is $T_1$ (because a space is $T_1$ iff singletons are closed). Alas and Wilson also showed:

**Theorem D.** *Every compact space of cardinality $\leq \mathfrak{t}$ is sequentially compact.*

**Theorem E.** *Every countably compact space of hereditary Lindelöf degree $< \mathfrak{t}$ is sequentially compact.*

In this paper we strengthen Theorem D and Theorem E with Theorems 1 and 4, respectively. But first, here is a quick corollary of Theorem E:

**Corollary 1.** *Every countably compact, hereditarily Lindelöf KC space is sequential.*

*Proof.* Every countably compact subset of a hereditarily Lindelöf space is compact. Since $\mathfrak{t}$ is uncountable, this corollary now follows from Theorem E and Theorem F below, whose proof is implicit in [IN, Proposition 1.19].

**Theorem F.** *A countably compact KC space is sequential iff it is sequentially compact and C-closed.*

Recall that space is called **C-closed** if every countably compact subspace is closed. Despite the similarity in the definitions, being C-closed is much more restrictive than being KC, as Theorem F indicates (sequentially compact, non-sequential compact $T_2$ spaces abound).

The following example, standard in the study of sequential spaces and their subspaces, shows how "KC" cannot be eliminated from the above corollary.

**Example 1.** Let $p$ be a point of $\beta\omega \setminus \omega$ and let $X$ be the space obtained by adding a point $x$ to $\omega \cup \{p\}$ and declaring the neighborhoods of $x$ to be those subsets of $X$ that contain $x$ and all but finitely many points of $\omega$. Being countable, $X$ is hereditarily Lindelöf, and $X$ is countably compact, but it is not sequential since no sequence from $X \setminus \{p\}$ converges to $p$.

## 2. Net weight in compact spaces

Recall that a network is defined like a base, except that its members are not required to be open: $\mathfrak{N}$ is a *network for* $X$ if for every neighborhood $W$ of a point $x$, there exists $N \in \mathfrak{N}$ such that $x \in N \in W$. The **net weight** of a space is the least cardinality of a network for it. Obviously, hereditary Lindelöf degree $\leq$ net weight $\leq$ weight, and for compact Hausdorff spaces, net weight equals weight. However, this equality breaks down for more general spaces. Example 1 is a countable $T_1$ space which can be made to be of weight $\mathfrak{c}$, and the following example is even a KC space.

**Example 2.** Let $X$ be the one-point compactification of the space $\mathbb{Q}$ of rational numbers. The singletons of $X$ form a countable network, but since $\mathbb{Q}$ has a copy of every countable ordinal in it, the extra point $\infty$ of $X$ does not have a countable local base. Of course, $X$ is not Hausdorff, but it is a KC space: a subset of $\mathbb{Q}$ is closed in $X$ iff it is compact, while subsets containing $\infty$ are easily shown to be compact iff their traces on $\mathbb{Q}$ are closed.

The cofinality of the collection of compact subsets of $\mathbb{Q}$ is the dominating number $\mathfrak{d}$ [vD, Theorem 8.7]. The proof is long and intricate, except for the cofinality being $\geq \mathfrak{d}$. It follows that the weight of the one-point compactification of $\mathbb{Q}$ is $\mathfrak{d}$ also.

The following theorem includes ideas from the proofs of both Theorem D and Theorem E. It clearly implies Theorem D (just let $\mathfrak{N}$ be the set of singletons).

**Theorem 1.** *If $X$ is a compact space with a network $\mathfrak{N}$ of cardinality $\leq \mathfrak{t}$, such that every point of $X$ is in fewer than $\mathfrak{t}$ members of $\mathfrak{N}$, then $X$ is sequentially compact.*

*Proof.* Let $\mathfrak{N} = \{N_\alpha : \alpha < \kappa\}$ be a network for $X$ with $\kappa \leq \mathfrak{t}$. Let $\langle x_n : n \in \omega\rangle$ be a sequence in $X$. Let $A_{-1} = \omega$. Suppose $\alpha < \kappa$ and we have an $\alpha$-sequence of infinite subsets $\langle A_\beta : -1 \leq \beta < \alpha\rangle$ of $\omega$ such that $A_\gamma \subseteq^* A_\beta$ whenever $\beta < \gamma$. [Note the order reversal. As usual, $A \subseteq^* B$ means $A \setminus B$ is finite.] Using the fact that $\alpha < \mathfrak{t}$, let $A'_\alpha \subseteq^* A_\beta$ for all $\beta < \alpha$. If there is an infinite subset $A \subset A'_\alpha$ and an open set $U$ containing $N_\alpha$ such that $\{x_n : n \in A\}$ misses $U$, let $U_\alpha = U$ and $A_\alpha = A$; otherwise, let $U_\alpha = \emptyset$ and let $A_\alpha = A'_\alpha$.

For each $x \in X$ let $\alpha(x) < \mathfrak{t}$ be such that all members of $\mathfrak{N}$ containing $x$ are indexed before $\alpha(x)$. If $\kappa = \mathfrak{t}$ and $\langle x_n : n \in A_{\alpha(x)}\rangle$ does not converge to $x$, then there exists $\beta(x) < \alpha(x)$ such that $x \in N_{\beta(x)} \subset U_{\beta(x)}$ and so $U_{\beta(x)}$ contains $x_n$ for only finitely many $n \in A_{\alpha(x)}$. If $\kappa < \mathfrak{t}$ let $A_{\alpha(x)}$ be some infinite set almost contained in every $A_\alpha$ and choose $\beta(x)$ as before. If $\langle x_n : n \in \omega\rangle$ has no convergent sequences then the sets $U_{\beta(x)}$ are an open cover of $X$. Let $U_{\beta(y_1)}, \ldots, U_{\beta(y_n)}$ be a finite subcover with the $\beta(y_i)$ in ascending order. Then there exists $i$ such that $x_k \in U_{\beta(y_i)}$ for infinitely many $k$ in $A_{\beta(y_n)}$, contradicting the way $A_{\beta(y_i)}$ was defined. □

Lemma 1 below actually has Theorem 1 as a corollary, but the first half of its proof is "nonlinear" and we thought the simpler, "linear" proof of Theorem 1 was worth presenting.

**Lemma 1.** *Let $\kappa$ be a regular cardinal and assume that every splitting tree has a chain of cardinality $\kappa$. If $X$ is a compact space with a network $\mathfrak{N}$ of cardinality $\kappa$, such that every point of $X$ is in fewer than $\kappa$ members of $\mathfrak{N}$, then $X$ is sequentially compact.*

*Proof.* Let $\mathfrak{N} = \{N_\alpha : \alpha < \kappa\}$. Toward a contradiction, let $\langle x_n : n \in \omega\rangle$ be a sequence in $X$ with no convergent subsequence. Let $\mathcal{A}_{-1} = \{\omega\}$. Suppose $\alpha < \kappa$

and we have an $\alpha$-sequence of MAD families $\langle \mathcal{A}_\beta : -1 \leq \beta < \alpha \rangle$ on $\omega$ such that $\mathcal{A}_\gamma$ refines $\mathcal{A}_\beta$ whenever $\beta < \gamma$. Since $\alpha < \mathfrak{h}$, there exists a MAD family $\mathcal{A}'_\alpha$ which refines $\mathcal{A}_\beta$ for all $\beta < \alpha$.

Let us denote by $\tau(N_\alpha)$ the collection of all open sets $U$ such that $N_\alpha \subset U$. For a given $B \in \mathcal{A}'_\alpha$, let $\mathcal{A}(B) \subset [B]^\omega$ be an almost disjoint family maximal with respect to the property $C \in [B]^\omega$ and the set $\{x_n : n \in C\}$ is either almost contained in each member of $\tau(N_\alpha)$ or almost disjoint from some member of $\tau(N_\alpha)$. Then, put $\mathcal{A}_\alpha = \bigcup\{\mathcal{A}(B) : B \in \mathcal{A}'_\alpha\}$. It is easy to realize that $\mathcal{A}_\alpha$ is actually a MAD family on $\omega$. Now, thanks to our hypothesis, we may fix a chain $\{A_\alpha : \alpha < \kappa\}$ in the tree $\bigcup\{\mathcal{A}_\alpha : \alpha < \kappa\}$.

The remainder of the proof is the same as in Theorem 1, except that we have not yet defined $U_\alpha$. Now that we have the chain in hand, let $U_\alpha$ be an open set containing $N_\alpha$ and missing $\{x_n : n \in A_\alpha\}$ unless every open set containing $N_\alpha$ meets $\{x_n : n \in A_\alpha\}$ (hence almost contains it), in which case we let $U_\alpha = \emptyset$. And now we follow the second paragraph of the proof of Theorem 1. □

Theorem 1 is the case $\kappa = \mathfrak{t}$ of Lemma 1: every branch of a splitting tree is a complete tower, so every splitting tree has a chain of length $\mathfrak{t}$. The case $\kappa = \mathfrak{h}$ of Lemma 1 is also of interest:

**Theorem 2.** *If every splitting tree has a chain of length $\mathfrak{h}$, then a compact space $X$ with a network $\mathfrak{N}$ of cardinality $\leq \mathfrak{h}$, such that every point of $X$ is in fewer than $\mathfrak{h}$ members of $\mathfrak{N}$, is sequentially compact.* □

**Corollary 2.** *If every splitting tree has a chain of length $\mathfrak{h}$, then every compact space of cardinality $\leq \mathfrak{h}$ is sequentially compact.* □

Corollary 2 only improves on Theorem B+ if $2^{\mathfrak{t}} = \mathfrak{c} = \mathfrak{h}$ (obviously, $2^{\mathfrak{t}} \geq \mathfrak{c} \geq \mathfrak{h}$ in general) but there do exist such models, as we shall see later [Example 6].

The hypothesis in Theorem 2 and Corollary 2 is equivalent to saying that every splitting tree of height $\mathfrak{h}$ "has long branches," as the jargon goes—a "long branch" being one that meets every level of the tree. We will see later [Example 5] that this hypothesis cannot be eliminated even for compact Hausdorff spaces.

We also cannot remove the restriction on the order of $\mathfrak{N}$ at each point in Theorems 1 and 2: in any model of $\mathfrak{t} = \mathfrak{s}$, the space $2^{\mathfrak{t}}$ is not sequentially compact; but it has a base (hence a network) of cardinality $\mathfrak{t} = \mathfrak{h}$. We do not know whether it is even consistent to remove the order restriction from Theorem 2:

**Problem 2.** Is it consistent that every compact space of net weight $\mathfrak{h}$ is sequentially compact?

However, in models of $\mathfrak{t} < \mathfrak{h}$, we can remove this restriction in Theorem 1 and even weaken "compact" to "countably compact". In fact, there are basically only two cases of Lemma 1: the case $\kappa = \mathfrak{h}$ of Theorem 2, and the case $\kappa < \mathfrak{h}$ where we get sequential compactness outright; this is the gist of one direction in the second of two topological characterizations

of $\mathfrak{h}$ in the following section.

## 3. Countably compact spaces and the cardinal $\mathfrak{h}$

Our next theorem is a worthy companion for Theorems A and C of the Introduction, despite there being no mention of the compact case in it. As can be seen from Problem 2 above, we don't know whether the net weight characterization in the following theorem goes through for compact spaces. The cardinality characterization does fail, as will be seen in Section 4.

**Theorem 3.** $\mathfrak{h}$ = *the least cardinality of a countably compact space which is not sequentially compact* = *the least net weight of a countably compact space which is not sequentially compact. The examples include initially* $<\mathfrak{h}$*-compact KC spaces.*

*Proof.* Since net weight $\leq$ cardinality, it is enough to prove that the least net weight $\geq \mathfrak{h}$, and that the least cardinality is $\leq \mathfrak{h}$. To prove $nw \geq \mathfrak{h}$, let $X$ be any space with a network $\mathfrak{N}$ of size $\kappa$, where $\kappa < \mathfrak{h}$, and suppose $X$ is not sequentially compact. Let $\sigma = \langle x_n : n \in \omega \rangle$ be a sequence in $X$ without a convergent subsequence. We will show $X$ is not countably compact.

We begin with the first paragraph in the proof of Lemma 1, except that, at the very end, we pick our chain $\{A_\alpha : \alpha < \kappa\}$ in the tree $\bigcup\{\mathcal{A}_\alpha : \alpha < \kappa\}$ so that there is an infinite set $C$ almost contained in every $A_\alpha$. This we can do because our tree is not splitting. We claim that $\{x_n : n \in C\}$ does not have a complete accumulation point.

For each $x \in X$, there exists an open set $V$ containing $x$ that omits infinitely many points of $\{x_n : n \in C\}$ and hence of every $\{x_n : n \in A_\alpha\}$. Pick $\beta < \kappa$ such that $N_\beta \subset V$; then, by the way $A_\beta$ was chosen, it is almost disjoint from some open set $U$ which contains $N_\beta$ and with it $x$. So $x$ is not a complete accumulation point of $\{x_n : n \in C\}$. $\square$

The $\kappa = \mathfrak{h}$ case of the following example completes the proof of Theorem 3.

**Example 3.** Let $\mathbb{N}$ be the set of positive integers, defined in such a way as to be disjoint from the class of ordinals. Let $\kappa$ be the height of some splitting tree $\mathcal{T} = \bigcup\{\mathfrak{M}_\alpha : \alpha < \kappa\}$ on $\mathbb{N}$ where each $\mathfrak{M}_\alpha$ is an infinite MAD family on $\mathbb{N}$ and $\mathfrak{M}_\alpha$ refines $\mathfrak{M}_\beta$ whenever $\beta < \alpha$. Let $X$ have $\mathbb{N} \cup \kappa$ as an underlying set. We will define the topology on $X$ with the help of $\mathcal{T}$.

Points of $\mathbb{N}$ are isolated. If $\alpha, \beta \in \kappa \cup \{-1\}$ let $(\beta, \alpha] = \{\xi : \beta < \xi \leq \alpha$. Let a base for the neighborhoods of $\alpha$ be all sets of the form

$$N(\alpha, \beta, \mathcal{F}, F) = (\beta, \alpha] \cup \mathbb{N} \setminus \left(\bigcup \mathcal{F} \cup F\right)$$

such that $\beta < \alpha$, and $\mathcal{F}$ is a finite subcollection of $M_\alpha$ and $F$ is a finite subset of $\mathbb{N}$. Note that if $\alpha$ is either 0 or a successor, it is enough to take the collection of all $N(\alpha, \alpha-1, \mathcal{F}, F)$.

**Claim 1.** *This defines a topology.*

**Claim 2.** *$X$ is a KC space.*

**Claim 3.** *$X$ is countably compact.*

**Claim 4.** *If $\kappa$ is regular (in particular, if $\kappa = \mathfrak{h}$), then $X$ is initially $<\mathfrak{h}$-compact.*

**Claim 5.** *$\mathbb{N}$ is dense in $X$, but no nontrivial sequence from $\mathbb{N}$ converges to any point of $X$.* [In other words, every sequence in $\mathbb{N}$ that converges in $X$ is eventually constant.]

*Proof of Claim 1.* We need only show that if $\xi$ is an ordinal in $N(\alpha, \beta, \mathcal{F}, s) \cap N(\alpha', \beta', \mathcal{F}', s')$, then there is a basic neighborhood of $\xi$ in the intersection. Let $\beta''$ be the maximum of $\beta, \beta'$. Then $(\beta'', \xi]$ is a subset of both $(\beta, \alpha]$ and $(\beta', \alpha']$. Moreover, for each $F \in \mathcal{F} \cup \mathcal{F}'$ there is a unique $M_F \in \mathfrak{M}_\xi$ such that $F \subset^* M_F$. Let $s_F = F \setminus M_F$ and let $s'' = s \cup s' \cup \bigcup\{s_F : F \in \mathcal{F} \cup \mathcal{F}'\}$. it is easy to see that $N(\xi, \beta'', \mathcal{F} \cup \mathcal{F}', s'')$ is the desired basic neighborhood of $\xi$. □

The foregoing proof also essentially shows that each point $\xi$ of $\kappa$ has a local base of sets of the form $N(\xi, \eta, \mathcal{F}, s)$.

*Proof of Claim 2.* We show that every compact subset of $X$ meets $\mathbb{N}$ in a finite set. Since the relative topology on $\kappa$ is the usual (Hausdorff) order topology and $\kappa$ is closed, Claim 2 follows.

Let $K$ be a compact subset of $X$. Then $K \cap \kappa$ is compact in $\mathfrak{h}$, hence has a greatest element $\alpha$. Suppose $K \cap \mathbb{N}$ is infinite. Let $M \in \mathfrak{M}_\alpha$ hit $K$; then $\{N(\alpha, -1, \{M\}, \emptyset)\} \cup \{\{n\} : n \in \mathbb{N}\}$ is an open cover of $K$ without a finite subcover, contradicting compactness of $K$. □

*Proof of Claim 3.* Let $A$ be an infinite subset of $\mathbb{N}$. Since $\mathcal{T}$ is splitting, there exists $\alpha$ such that infinitely many members of $\mathfrak{M}_\alpha$ hit $A$. Indeed, choose $\alpha_0$ such that at least two members of $\mathfrak{M}_{\alpha_0}$ hit $A$ and let $\mathcal{N}_0$ be the collection of all infinite sets of the form $M \cap A$, $M \in \mathfrak{M}_{\alpha_0}$. If $\mathcal{N}_0$ is finite, choose $\alpha_1 > \alpha_0$ so that every member of $\mathcal{N}_0$ is hit by at least two members of $M_{\alpha_1}$ and define $\mathcal{N}_1$ analogously to $\mathcal{N}_0$. Choosing $\alpha_n$ in this fashion for all $n \in \omega$ if necessary, let $\alpha$ be their supremum; then $\alpha$ is as desired. It follows that $\alpha$ is a cluster point of $A$. □

*Proof of Claim 4.* A space defined to be initially $<\lambda$-compact if every open cover of cardinality $< \lambda$ has a finite subcover. Thanks to Claim 3, it is enough to show that every open cover of $X$ of cardinality $< \kappa$ has a countable subcover. And this is immediate from the fact that the subspace $\kappa$ has this property. □

*Proof of Claim 5.* $\mathbb{N}$ is dense because each $\mathfrak{M}_\alpha$ is infinite. To show the rest of Claim 5, let $A$ be an infinite subset of $\mathbb{N}$ and let $\alpha$ be as in the proof of Claim 3. If $\alpha < \xi < \kappa$, then infinitely many members of $\xi$ hit $A$, so the closure of $A$ includes a terminal segment of $\kappa$. But in a KC space, convergent sequences have unique limits. So no 1-1 sequence from $\mathbb{N}$ converges, and this is easily seen equivalent to only trivial sequences converging. □

The one-point compactification of Example 3 does not settle Problem 2: every infinite subset of $\mathbb{N}$ converges to the extra point.

Theorem 3 is not optimal for Hausdorff spaces: if we let $\mu_0$ stand for the least cardinality of a countably compact $T_2$ space that is not sequentially compact, then $\mathfrak{s} \leq \mu_0$ as shown in Theorem 4.1 of [BvDMW]. The following problem from that paper is still unsolved as far as we know.

**Problem 3.** Is $\mu_0 = \mathfrak{s}$?

Let $\mu_1$ stand for the least cardinality of an infinite, countably compact $T_2$ space in which every convergent sequence is eventually constant. We now have $\mathfrak{s} \leq \mu_0 \leq \mu_1 \leq \mathfrak{c}$, the last inequality following from the existence of a countably compact subset of $\beta\omega$ of cardinality $\mathfrak{c}$. In [BvDMW], where $\mu_0$ is denoted $\mu_{ns}$ and $\mu_1$ is denoted $\mu_d$, it is shown that $\omega_1 = \mu_1 < \mathfrak{c}$ and $\omega_1 < \mu_0 = \mu_1 < \mathfrak{c}$ are both consistent, and the question is raised whether $\mu_0 = \mu_1$ is a theorem of ZFC; as far as we know, this is still unanswered.

Also, we do not know whether we can improve on $\mu_1$ by weakening the $T_2$ axiom to $T_1$. As pointed out in the introduction, nothing is gained by weakening it further.

Example 3 is so simple that all its standard cardinal invariants can be easily found, and (except for precaliber and caliber, which are proper classes rather than single cardinals) they are all either $\omega$ (density, $\pi$-weight, cellularity) or the supremum of the cardinals $< \kappa$ (pseudocharacter, tightness, hereditary $\pi$-character) or $\kappa$ (cardinality, net weight, spread, hereditary density, [hereditary] Lindelöf degree, hereditary $\pi$-weight) or $\mathfrak{c}$ (character, weight, the number $RO(X)$ of regular open sets) or, for $o(X)$ (the number of open sets), $2^\kappa$. To see this last fact, note that the $o(X) \leq 2^{|X|}$ for any space, while the successor ordinals form a discrete subspace $D$ of cardinality $\kappa$, and the sets $A \cup \mathbb{N}$ are open, and distinct for distinct $A \subset D$.

Next we strengthen Theorem E in some models.

**Theorem 4.** *Let $X$ be a countably compact space. If every splitting tree has a chain of cardinality $hL(X)^+$, then $X$ is sequentially compact.*

*Proof.* In what follows, $S^\bullet$ denotes the set of all complete accumulation points of $S$. Note that $S^\bullet$ is closed in $X$ and that $S^\bullet \subset R^\bullet$ whenever $S$ is almost contained in $R$.

Let us assume by contradiction that there exists a set $A \in [X]^\omega$ with no non-trivial convergent subsequence. This means that for any $B \in [A]^\omega$ and any $x \in \overline{B}$ there exists some open set $U_x$ such that $x \in U_x$ and $C = B \setminus U_x$ is infinite. By countable compactness, $\overline{C}$ is a proper subset of $\overline{B}$, and $C^\bullet$ is a proper subset of $B^\bullet$.

For $\alpha < \mathfrak{h}$ let us suppose to have already defined a collection $\{\mathcal{A}_\gamma : \gamma < \alpha\}$ of MAD families contained in $[A]^\omega$ satisfying:

(1) if $\beta < \gamma < \alpha$ then $\mathcal{A}_\gamma$ "strongly refines" $\mathcal{A}_\beta$, *i.e.*, each member $C \in \mathcal{A}_\gamma$ is almost contained in some $B \in \mathcal{A}_\beta$ and $C^\bullet$ is a proper subset of $B^\bullet$.

If $\alpha = \beta + 1$ and $\mathcal{A}_\beta$ has been defined, then for each $B \in \mathcal{A}_\beta$ we let $\mathcal{E}(B) \subset [B]^\omega$ be an almost disjoint family maximal with respect to the property that $C^\bullet$ is a proper subset of $B^\bullet$ for any $C \in \mathcal{E}(B)$. Put $\mathcal{A}_\alpha = \bigcup\{\mathcal{E}(B) : B \in \mathcal{A}_\beta$. Using countable compactness of $X$, it is easy to check that $\mathcal{A}_\alpha$ is a MAD family on $A$.

If $\alpha$ is a limit ordinal then, in order to define $\mathcal{A}_\alpha$, observe first that, as $|\alpha| < \mathfrak{h}$, there exists an infinite subset $S$ of $A$ which is almost contained in some (unique) member of $\mathcal{A}_\gamma$ for each $\gamma < \alpha$. Let $\mathcal{S}$ be the collection of all such $S$ and let $\mathcal{A}_\alpha$ be a maximal almost disjoint family of members of $\mathcal{S}$. By the induction hypothesis, $\mathcal{A}_\alpha$ strongly refines all $\mathcal{A}_\beta$, $\beta < \alpha$. It is also a MAD family on $A$: for any $B \in [A]^\omega$ the trace of the tree $\{\mathcal{A}_\gamma : \gamma < \alpha\}$ on $B$ cannot be splitting and so there exist infinite subsets of $B$ in $\mathcal{S}$.

The tree $\bigcup\{\mathcal{A}_\alpha : \alpha < \mathfrak{h}\}$ has a chain $\mathcal{C}$ of cardinality $hL(X)^+$. As $hL(X)^+ \leq \mathfrak{h}$, this is obvious if the tree is not splitting and follows from our hypothesis in the other case. Then the family $\{C^\bullet : C \in \mathcal{C}\}$ is a strictly decreasing collection of closed sets, in contrast with the definition of $hL(X)$. □

Theorem E is a corollary of Theorem 4, just as Theorem 1 is a corollary of Lemma 1. We also have:

**Corollary 3.** *If $\kappa < \mathfrak{h}$, then every countably compact space of hereditary Lindelöf degree $< \kappa$ is sequentially compact.* □

**Corollary 4.** *If every splitting tree has a chain of length $\mathfrak{h}$, or if $\mathfrak{h}$ is a limit cardinal, then $\mathfrak{h} = min\{hL(X) : X$ is countably compact but not sequentially compact$\}$.*

*Proof.* Example 3, with $\kappa = \mathfrak{h}$, provides a countably compact KC non sequentially compact space $X$ satisfying $hL(X) = |X| = \mathfrak{h}$. Now apply Theorem 4 to conclude that every countably compact space $X$ satisfying $hL(X) < \mathfrak{h}$ is sequentially compact. □

Since $\mathfrak{h}$ is clearly regular, saying that it is a limit cardinal, as in Corollary 4, is the same as saying that it is weakly inaccessible. But do we need to say anything at all here? We mean:

**Problem 4.** If $hL(X) < \mathfrak{h}$, is $X$ sequentially compact if it is **(a)** countably compact or **(b)** compact?

## 4. Compact spaces and the Novak number

We return to compact spaces with our next theorem, which uses the following concept.

The Novak (or Baire) number $\mathfrak{n}$ of $\omega^*(=\beta\omega-\omega)$ is the smallest cardinality of a cover of $\omega^*$ by nowhere dense sets. A good reference for this cardinal is again [BPS], where it is denoted $n(N^*)$. We recall that $\max\{\mathfrak{t}^+,\mathfrak{h}\} \leq \mathfrak{n} \leq 2^{\mathfrak{c}}$ and the equality $\mathfrak{h}=\mathfrak{n}$ holds if and only if there is a splitting tree of height $\mathfrak{h}$ without long chains (see [BPS], 2.10 and 3.4).

**Theorem 5.** *Let $X$ be a compact space. If $|X| < \mathfrak{n}$ then $X$ is sequentially compact.*

*Proof.* Assume by contradiction that $X$ is not sequentially compact and let $S = \langle x_n : n < \omega\rangle$ be a sequence with no convergent subsequence. For any $x \in X$ let $\mathcal{A}_x$ be the collection of all $A \in [\omega]^\omega$ such that there exists an open neighbourhood $U$ of $x$ satisfying $x_n \notin U$ for each $n \in A$. Fix a maximal almost disjoint subcollection $\mathcal{B}_x \subset \mathcal{A}_x$. As we are assuming that $S$ does not have any subsequence converging to $x$, it follows that $\mathcal{B}_x$ is actually a MAD family on $\omega$. The set $N_x = \omega^* \setminus \bigcup\{B^* : B \in \mathcal{B}_x\}$ is nowhere dense and therefore we may pick a point $p \in \omega^* \setminus \bigcup\{N_x : x \in X\}$. [As usual, $B^*$ denotes the Stone-Čech remainder $c\ell_{\beta\omega} B \setminus B$.] This means that $p \in B_x^*$ for some $B_x \in \mathcal{B}_x$ and any $x \in X$. Consequently, the family $\{B_x : x \in X\}$ has the finite intersection property and the compactness of $X$ ensures the existence of a point $z \in \bigcap\{\overline{\{x_n : n \in B_x\}} : x \in X\}$. But $B_z \in \mathcal{A}_z$ and so $z \notin \overline{\{x_n : n \in B_z\}}$, according to the way $\mathcal{A}_z$ was defined. This contradiction proves the theorem. □

Notice that a version of Example 3 where $\kappa = \mathfrak{h}$, in a model where all splitting trees have long chains, shows that consistently Theorem 5 cannot be extended to countably compact (even KC) spaces.

Theorem 5 is evidently an improvement of both Theorem D and Corollary 2. We do not know whether the formula in Theorem 5 is optimal—*i.e.,* whether there always exists a compact space of cardinality $\mathfrak{n}$ which is not sequentially compact. However, such a space (if there is any) cannot always be Hausdorff or even KC. This follows from Theorem B+ and Theorem 5 and the fact that there are models where $\mathfrak{n} < 2^{\mathfrak{t}}$ (for instance, if $\mathfrak{c}$ is a singular cardinal then 3.5 and 4.2 in [BPS] give $\mathfrak{h} \leq cf(\mathfrak{c})$ and $\mathfrak{n} \leq \mathfrak{h}^+$, so that $\mathfrak{n} < \mathfrak{c} \leq 2^{\mathfrak{t}}$). See also Example 7 below.

The more interesting question of the consistency of $2^{\mathfrak{t}} < \mathfrak{n}$ will be treated in Example 6.

**Problem 5.** Let $\mu_2$ be the least cardinality of a compact $T_2$ space that is not sequentially compact. Is $\mu_2$ also equal to the least cardinality of **(a)** a compact space that is not sequentially compact? **(b)** a compact KC space that is not sequentially compact?

Clearly, if Problem 5(a) has an affirmative answer, so does Problem 1.

The results discussed here give $\max\{2^{\mathfrak{t}}, \mathfrak{n}\} \leq \mu_2 \leq 2^{\mathfrak{h}}$. The latter inequality follows from, and is further improved by, a general construction which we next recall.

**Example 4.** Given any splitting tree $T$, there is a is a compact Hausdorff space $\Phi$ with underlying set $\omega \cup T^*$ where $T^*$ is the order completion of $T$, in which $\omega$ is a dense set of isolated points and no sequence on $\omega$ can converge to any point of $T^*$. The construction of $\Phi$ is given in several places, including [Ny]. The relative topology on $T^*$ is the coarse wedge topology. This is the topology which has as a subbase all sets of the form $V_x = \{t \in T^* : t \geq x\}$ and their complements, where $x$ is on a successor level.

The hereditary Lindelöf degree of $\Phi$ is $\mathfrak{c}$, because the points on level $\omega + 1$ of $T$ are a discrete subspace of cardinality $\mathfrak{c}$.

Each infinite level of $T$ is of cardinality $\mathfrak{c}$, so the cardinality of $\Phi$ is the number of branches of $T$. Thus if we let $\mu_3$ be the least number of branches in a splitting tree, we can improve the inequality $\mu_2 \leq 2^{\mathfrak{h}}$ to $\mu_2 \leq \mu_3 \leq 2^{\mathfrak{h}}$. The latter inequality can be strict, as in Example 5 below.

We can get the points on successor levels to be of character $\mathfrak{a}$ in $\Phi$, where $\mathfrak{a}$ is the least cardinality of an infinite MAD family on $\omega$, but we do not know whether the character of $\Phi$ itself (defined, as usual, as the supremum of the characters of its points) can be less than $\mathfrak{c}$.

**Problem 6.** What is the least character **(a)** of a countably compact space that is not sequentially compact? **(b)** of a compact space that is not sequentially compact?

Since character $\leq$ weight, this character must be $\leq \mathfrak{s}$ and it is also $\geq \mathfrak{p}$: if a cluster point of a sequence has a local base of cardinality $< \mathfrak{p}$, then there is a subsequence converging to it. This is obvious from the definition of $\mathfrak{p}$ : it is the least cardinal such that there is a base $\mathcal{B}$ for a free filter on $\omega$ for which there is no infinite subset of $\omega$ almost contained in every member of $\mathcal{B}$.

Obviously $\mathfrak{p} \leq \mathfrak{t}$, but it is still an unsolved problem whether $\mathfrak{p} = \mathfrak{t}$.

**Example 5.** Dordal [D1] showed that in the Mathias model, $\mathfrak{t} = \omega_1$ and $\mathfrak{h} = \omega_2 = 2^{\omega} = 2^{\omega_1}$ and there is a splitting tree $T$ of height $\mathfrak{h}$ that has no long branches; hence $\mathfrak{n} = \mathfrak{h}$. Dow [D3] constructed other forcing models with splitting $T$ having the same properties. If we use these $T$ to construct $\Phi$ then $|\Phi| = |T^*| = \omega_2 = \mathfrak{h}$, whence the hypothesis in Corollary 2 cannot be eliminated. We do not know whether it is needed for Corollary 4 (see Problem 4).

**Example 6.** Dordal [D2] has shown that if one begins with a model of GCH, then obtains a model of $MA + \mathfrak{c} = \kappa$ by ccc forcing, and then forces with what was the full binary tree of height $\omega_1$ in the *original* model, then one obtains a model of $\mathfrak{h} = \mathfrak{c} = \kappa$ and $\mathfrak{t} = \omega_1$. [As seen by the ground model, this is just forcing by the product $\mathbf{P}\times\mathbf{Q}$ where $\mathbf{P}$ is the ccc forcing and $\mathbf{Q}$ is the full binary tree of height $\omega_1$.] Since the second forcing poset is of cardinality $\omega_1$, it does not increase the value of either $2^\omega$ or $2^{\omega_1}$, and Dordal has shown that no cardinals are collapsed by it. [This is the tricky detail, and would not work if the first forcing were not ccc.] So in the final model, $2^\mathfrak{t} = \mathfrak{c} = \mathfrak{h}$.

On the other hand, it is easy to see from the Lemma in Example V of [BPS] that if $\kappa = \omega_2$ then in the final model, $\mathfrak{n} \geq \omega_3$, so that every compact Hausdorff space of cardinality $2^\mathfrak{t}$ is sequentially compact. This gives a negative solution to Question 6.6 in [vD].

**Example 7.** Back in 1967, Hechler introduced nonlinear iterated forcing at a conference [He] where he showed that there are ccc forcing posets that give $({}^\omega\omega, <^*)$ a cofinal family of increasing functions of any "reasonable" order type.

If we use one that gives a cofinal family of order type $\omega_2 \times \omega_1$, we can use the family to construct a pair $(\mathcal{A}, \mathcal{B})$ of totally ordered families of nowhere dense subsets of $\omega^*$ whose union is all of $\omega^*$, with $\mathcal{A}$ of order type $\omega_1$ and $\mathcal{B}$ of order type $\omega_2$ [NV]. [Also, $\bigcup \mathcal{A}$ is disjoint from $\bigcup \mathcal{B}$.] Then $\mathfrak{n} = \omega_2$ in this model, the smallest value possible for $\mathfrak{n}$. Since the forcing is ccc, we can also make $\mathfrak{c}$, and hence $2^{\omega_1}$, "arbitrarily large" and either regular or singular, by choosing our ground model that way.

Despite its simplicity, this family of nowhere dense sets is ill-suited for producing a compact space of cardinality $< 2^{\omega_1}$ that is not sequentially compact. If we simply take the quotient space of $\beta\omega$ associated with the partition generated by $\mathcal{A} \cup \mathcal{B}$, then every infinite subset of $\omega$ converges to every point outside $\omega$.

## 5. Countably compactness and special bases

Our remaining theorems involve the concept of an $\omega$-in-countable base and a generalization.

**Definition.** Let $\kappa$ and $\lambda$ be cardinals. A collection $\mathcal{B}$ of sets is said to be *$\kappa$-in-$\leq \lambda$ [resp. $\kappa$-in-$<\lambda$]* if every set of cardinality $\kappa$ is contained in no more than [*resp.* fewer than] $\lambda$ members of $\mathcal{B}$. $\kappa$-in-$\omega$ is referred to as *$\kappa$-in-countable.*

Countably compact spaces have a strange dual personality where $\omega$-in-countable bases are concerned: they are very well behaved in some models of ZFC and can behave very badly in others. The following theorems from [BG] illustrate this. The first uses the concept of an HFD (hereditarily finally dense) subset of $2^{\omega_1}$. These spaces exist under the continuum hypothesis (CH) and are countably compact and Tychonoff (in fact, hereditarily normal), but neither compact nor sequentially compact.

**Theorem G.** *Every HFD subset of $2^{\omega_1}$ has an $\omega$-in-countable base.*

**Theorem H.** $[\mathfrak{p} > \omega_1]$ *Every countably compact Hausdorff space with an $\omega$-in-countable base is metrizable.*

**Theorem Z.** *Every compact Hausdorff space with an $\omega$-in-countable base is metrizable.*

The natural generalization of "compact metrizable" to spaces that are not necessarily Hausdorff is "compact second countable." So the equivalence of (1) and (2) in the following theorem comprises a twofold strengthening of Theorem H:

**Theorem 6.** *The following statements are equivalent.*

(1) $\mathfrak{p} > \omega_1$
(2) *Every countably compact $T_1$ space with an $\omega$-in-countable base is second countable (hence compact).*
(3) *Every compact $T_1$ space with an $\omega$-in-countable base is second countable.*
(4) *Every compact $T_1$ space with an $\omega$-in-countable base is sequential.*

Other equivalent statements of note are $\mathfrak{t} > \omega_1$ [vD, Theorem 3.1] and $MA_{\omega_1}$($\sigma$-centered) [W, Theorem 5.16].

To show Theorem 6, we first show two lemmas whose combined proof is very similar to that of [BG, Theorem 4.4]:

**Lemma 2.** $[\mathfrak{p} > \omega_1]$ *Let $X$ be a topological space with an $\omega$-in-countable base $\mathcal{B}$. For each $x$ in $X$ let $\mathcal{B}(x) = \{B \in \mathcal{B} : x \in B\}$. Let*

$$Y = \{x \in X : x \text{ is in the closure of a countable subset of } X \setminus \{x\}\}$$

*Then $\mathcal{B}(y)$ is countable for all $y \in Y$.*

*Proof.* Assume indirectly that there is some $y \in Y$ such that $|\mathcal{B}(y)| > \omega$ and fix $\mathcal{C} \in [\mathcal{B}(y)]^{\omega_1}$. If $y$ is in the closure of the countable infinite set $S \subset X \setminus \{y\}$, then the trace of $\mathcal{C}$ on $S$ has the strong finite intersection property (i.e., every finite subcollection has infinite intersection), and $\mathfrak{p} > \omega_1$ ensures the existence of an infinite set $S_0 \subset S$ such that $S_0 \subset^* C$ for each $C \in \mathcal{C}$. This in turn implies the existence of an infinite set $S_1 \subset S_0$ which is contained in uncountably many elements of $\mathcal{C}$, in contrast with the hypothesis that $\mathcal{B}$ is $\omega$-in-countable. □

**Lemma 3.** $[\mathfrak{p} > \omega_1]$ *If $X$, $Y$, and $\mathcal{B}$ are as in Lemma 2, and $X$ is countably compact and $T_1$, then $\mathcal{B}(Y) = \bigcup\{\mathcal{B}(y) : y \in Y\}$ is countable and $Y$ is compact.*

*Proof.* Since $Y$ is countably compact and metaLindelöf, it is compact. Now by Mishchenko's lemma [E, 3.12.23 (f), p. 242], there are only countably many minimal open covers of $Y$ by members of $\mathcal{B}$, and the proof that $\mathcal{B}(Y)$ is countable is the same as the proof using Mishchenko's Lemma that every compact Hausdorff space with a point-countable base is metrizable [Ho1]. □

*Proof of Theorem 6.* To show (1) implies (2), let $X$, $Y$, and $\mathcal{B}$ be as in Lemma 2. Then every open subset of $X$ containing $Y$ is cofinite, otherwise countable compactness would give a countable set with a limit point in the complement, contradicting the definition of $Y$. Since $Y$ is compact, so is $X$. Also, $Y$ is a $G_\delta$: $\{x\}$ is closed for each point $x$ of $X \setminus Y$, so there is a minimal cover of $Y$ by members of $\mathcal{B}$ that misses $x$, and as noted in the proof of Lemma 3, there are only countably many such covers. So $X \setminus Y$ is countable. Hence, by the definition of $Y$, the points of $X \setminus Y$ are isolated in the relative topology of $X \setminus Y$. Now it is easy to see that $B(Y) \cup \{\{x\} : x \text{ is isolated in } X\}$ is a countable base for $X$.

Obviously, (2) implies (3) implies (4). We show (4) implies (1) in Theorem 6 by contrapositive, using $\mathfrak{t} = \omega_1$ to put a complete $\omega_1$-tower $\mathcal{T} = \{T_\alpha : \alpha < \omega_1\}$ on $\omega$, with 0 in every $T_\alpha$ and 1 in none of them. For a base we take $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$ where $\mathcal{B}_0$ is all cofinite subsets of $\omega$ and $\mathcal{B}_1$ is the collection of all sets of the form $\{0\} \cup A$ where $A$ is a cofinite subset of $T_\alpha$ for some $\alpha$.

This space is clearly compact and $T_1$, and it is not sequential because 0 is non-isolated but there is no sequence converging to it from the rest of the space. But $\mathcal{B}$ is $\omega$-in-countable because an infinite subset can be contained in at most countably many tower members, hence in only countably many members of $\mathcal{B}$ altogether. □

The following simple example shows that some separation beyond $T_0$ is needed in Theorem 6.

**Example 8.** Let $X = \omega_1 + 1$ with the topology whose nonempty open sets are the sets of the form $(\alpha, \omega_1]$. Clearly $X$ is $T_0$ and compact, and has a 2-in-countable, hence $\omega$-in-countable topology. However, it is neither second countable, nor first countable, nor sequential.

A more challenging theme is what happens if we try to improve the separation axiom in Theorem 6. For instance:

**Problem 7.** Is $\mathfrak{p} > \omega_1$ equivalent to the statement that every countably compact Hausdorff space with an $\omega$-in-countable base is **(a)** second countable? **(b)** first countable? **(c)** sequential?

Problem 7(a) is equivalent to asking for metrizability, thanks to Theorem H, even if we weaken "Hausdorff" to "KC": an easy proof by contrapositive shows that every first countable space in which convergent sequences have unique limits is Hausdorff. This suggests:

**Problem 8.** Is every countably compact KC space with an $\omega$-in-countable base Hausdorff?

**Problem 9.** Is $\mathfrak{p} > \omega_1$ equivalent to the statement that every countably compact $T_1$ space with an $\omega$-in-countable base is compact?

With "Hausdorff" in place of "$T_1$," Problem 9 becomes equivalent to Problem 7(a), thanks to Theorem Z. With "KC" in both problems, the two questions might be distinct, depending on how Problem 8 plays out.

Theorems G and H show that the statement "Every countably compact Hausdorff space with an $\omega$-in-countable base is compact" is ZFC-independent, so it is actual equivalence with $\mathfrak{p} > \omega_1$ that is the issue in Problems 7 and 9.

Theorem G also shows that some set-theoretic hypothesis is needed in our next theorem, where we return to the theme of sequential compactness.

**Lemma 4.** *Let $\mathcal{B}$ be a base for a countably compact space $X$ that has a sequence $\langle x_n : n \in \omega\rangle$ with no convergent subsequence. If $\mathcal{C}$ is a collection of infinite members of $\mathcal{B}$, and $\{x_n : n \in S\}$ is infinite and is almost contained in each member of $\mathcal{C}$, then there is a MAD family $\mathcal{M}$ on $S$ such that each set of the form $\{x_n : n \in M\}(M \in \mathcal{M})$ is contained in some $B_M \in \mathcal{B} \setminus \mathcal{C}$.*

*Proof.* Let $\mathcal{M}$ be an almost disjoint family of infinite subsets of $S$ which is maximal with respect to the property that $\{x_n : n \in M\}$ is contained in some $B_M \in \mathcal{B} \setminus \mathcal{C}$ for every $M \in \mathcal{M}$. Let us verify that $\mathcal{M}$ is actually a MAD family on $S$. Toward a contradiction, assume that there is an infinite set $T \subset S$ which is almost disjoint from every member of $\mathcal{M}$. As $X$ is countably compact, we may pick a complete accumulation point $x_T \in X$ of the set $\{x_n : n \in T\}$. By hypothesis the sequence $\langle x_n : n \in T\rangle$ does not converge to $x_T$ and so there exists some $B \in \mathcal{B}$ such that $x_T \in B$ and $\{x_n : n \in T\} \setminus B$ is infinite. Since the set $\{x_n : n \in S\}$ is almost contained in each member of $\mathcal{C}$, it immediately follows that $B \notin \mathcal{C}$. The infinite set $\{n : n \in T \text{ and } x_n \in B\}$ contradicts the maximality of $\mathcal{M}$ and the proof is complete. $\square$

**Theorem 7.** $[\mathfrak{h} > \omega_1]$ *Every countably compact space $X$ with an $\omega$-in-countable base $\mathcal{B}$ is sequentially compact.*

*Proof.* Toward a contradiction, let $\sigma = \langle x_n : n \in \omega\rangle$ be a sequence in $X$ without a convergent subsequence. For each infinite subset $A$ of $\omega$ let $\sigma(A) = \{x_n : n \in A\}$. According to Lemma 4, let $\mathfrak{M}_0$ be a maximal almost disjoint family of infinite subsets of $\omega$ such that $\sigma(M) \subset B_M \in \mathcal{B}$ for any $M \in \mathfrak{M}_0$.

Now suppose $\alpha < \omega_1$ and that a MAD family $\mathfrak{M}_\beta$ of subsets of $\omega$ has been defined for each $\beta < \alpha$, in such a way that if $\gamma < \beta < \alpha$ then $\mathfrak{M}_\beta$ refines $\mathfrak{M}_\gamma$. Furthermore, we assume that for every $M \in \bigcup\{\mathfrak{M}_\beta : \beta < \alpha\}$ the set $\sigma(M)$ is contained in some $B_M \in \mathcal{B}$ and if $M' \in \mathfrak{M}_\beta$, $M'' \in \mathfrak{M}_\gamma$ and $M' \subseteq^* M''$ then $B_{M'} \neq B_{M''}$. As $\alpha < \omega_1$, there exists a MAD family $\mathfrak{M}$ on $\omega$ which refines $\mathfrak{M}_\beta$ for each $\beta < \alpha$. Next, we may apply Lemma 4, with $S = M \in \mathfrak{M}$ and $\mathcal{C}_M = \{B_N : N \in \mathfrak{M}_\beta, M \subseteq^* N, \beta < \alpha\}$, to find a MAD family $\mathfrak{M}_\alpha(M)$ on $M$ in such a way that for any $N \in \mathfrak{M}_\alpha(M)$ the set

$\sigma(N)$ is contained in some $B_N \in \mathcal{B} \setminus \mathcal{C}_M$. Finally, let $\mathfrak{M}_\alpha = \bigcup\{\mathfrak{M}_\alpha(M) : M \in \mathfrak{M}\}$. As $\mathfrak{h} > \omega_1$, the tree $\mathcal{T} = \bigcup\{\mathfrak{M}_\alpha : \alpha < \omega_1\}$ is not splitting and so there exists some infinite set $S \subset \omega$ which is almost contained in some $M_\alpha \in \mathfrak{M}_\alpha$ for each $\alpha < \omega_1$. Letting $b_\alpha = B_{M_\alpha}$, the way the tree $\mathcal{T}$ was constructed implies that the family $\mathcal{D} = \{B_\alpha : \alpha < \omega_1\}$ consists of pairwise distinct elements and $\sigma(S) \subseteq^* B_\alpha$ for each $\alpha$. Now, we can easily find an infinite set $E \subset S$ and an uncountable $I \subset \omega_1$ in such a way that $\sigma(E) \subset B_\alpha$ for each $\alpha \in I$. This is obviously in contrast with the hypothesis that $\mathcal{B}$ is $\omega$-in-countable. □

A corollary of Theorem 7 and Theorem G is that $\mathfrak{h} > \omega_1$ is already enough to negate the existence of HFD subsets of $2^{\omega_1}$.

We do not know whether the set-theoretic hypothesis can be eliminated from the following theorem:

**Theorem 8.** $[\mathfrak{s} > \omega_1]$ *Every compact space $X$ with an $\omega$-in-countable base $\mathcal{B}$ is sequentially compact.*

*Proof.* Assume by contradiction that $X$ is not sequentially compact and let $A$ be the range of a sequence $\langle x_n : n \in \omega\rangle$ with no convergent subsequence. Denote by $\mathcal{B}'$ the collection of all $B \in \mathcal{B}$ which have a finite intersection with $A$. Let $\alpha \in \omega_1$ and let us assume to have chosen for each $\beta < \alpha$ a finite collection $\mathcal{C}_\beta \subset \mathcal{B}$ such that $\bigcup\{\mathcal{C}_\beta : \beta < \alpha\} \setminus \mathcal{B}'$ consists of pairwise distinct elements and $\overline{A} \subset \bigcup \mathcal{C}_\beta$ for each $\beta < \alpha$. Fix $x \in \overline{A}$. If $x \in \bigcup \mathcal{B}'$, then choose $B_x \in \mathcal{B}'$ such that $x \in B_x$. If $x \notin \bigcup \mathcal{B}'$, let $\mathcal{C}_x$ be the collection of all $B \in \bigcup\{\mathcal{C}_\beta : \beta < \alpha\}$ such that $x \in B$. As $\mathcal{C}_x$ is countable and the family $\{B \cap A : B \in \mathcal{C}_x\}$ has the strong finite intersection property, there exists an infinite set $A_x \subset A$ which is almost contained in each $B \in \mathcal{C}_x$. Since the sequence $A_x$ cannot converge to $x$, we may choose some $B_x \in \mathcal{B}$ in such a way that $x \in B_x$ and $A_x \setminus B_x$ is infinite. The latter condition implies in particular that $B_x \notin \mathcal{C}_x$. By compactness, finitely many of these $B_x$ covers $\overline{A}$ and we denote by $\mathcal{C}_\alpha$ such a collection. For any $\alpha \in \omega_1$ the set $A$ is almost contained in $\bigcup(\mathcal{C}_\alpha \setminus \mathcal{B}')$ and the family $\{B \cap A : B \in \bigcup\{\mathcal{C}_\alpha \setminus \mathcal{B}' : \alpha < \omega_1\}\}$ is not splitting on $A$. Therefore, there exists an infinite set $D \subset A$ and sets $C_\alpha \in \mathcal{C}_\alpha$ such that $D \subseteq^* C_\alpha$ for each $\alpha \in \omega_1$. The family $\{C_\alpha : \alpha \in \omega_1\}$ consists of pairwise distinct elements and we may finish as in the proof of Theorem 6. □

We close this section with some comments about generalizing the last three theorems to higher cardinals. We do not have a complete generalization of Theorem 6 because of the unsolved problem of whether $\mathfrak{p} = \mathfrak{t}$, and because towers are hard to replace by other filterbases in the proof that (4) implies (1). As it is, we have:

**Theorem 6a+.** *If $\kappa < \mathfrak{p}$ then every countably compact $T_1$ space with an $\omega$-in-$<\kappa$ base is second countable.*

The proof that (1) implies (2) in Theorem 6 goes through with a simple cardinal shift, including the substitution of "$\mathfrak{p}$-many" for "uncountably many" in the proof of Lemma 2, for which Szymanski's theorem that $\mathfrak{p}$ is a regular cardinal [vD, Theorem 3.1 (e)] is relevant.

**Theorem 6b+.** *If every countably compact $T_1$ space with an $\omega$-in-$<\kappa$ base is sequential, then $\kappa < \mathfrak{t}$.*

All it takes here is to replace the $\omega_1$-tower in the example with a complete $\lambda$-tower where $\lambda \leq \kappa$. The proof of Theorem 7 straightforwardly generalizes to give:

**Theorem 7+.** *If $\kappa < \mathfrak{h}$ then every countably compact space with an $\omega$-in-$<\kappa$ base is sequentially compact.*

In the proof of Theorem 8, there is a bottleneck in the definition of $B_x$ when $x \notin \bigcup \mathcal{B}'$, and the best we could do was:

**Theorem 8+.** *If $\kappa < \mathfrak{h}$ and $\kappa \leq \mathfrak{p}$, then every countably compact space with an $\omega$-in-$<\kappa$ base is sequentially compact.*

To show this, replace $\omega_1$ by $\mathfrak{p}$ and "countable" by "of cardinality $< \mathfrak{p}$" everywhere in the proof of Theorem 8.

## 6. Other cardinal invariants

In this final section, we return to the basic cardinal invariants, addressing both general themes mentioned in the introduction. To save space, we call a space **ponderous** if it is an infinite, countably compact space in which every convergent sequence is eventually constant. [As noted in the introduction, all such spaces are $T_1$.] This gives us four questions for each cardinal invariant, two for countably compact spaces in general and two for compact spaces, even without bringing in higher separation axioms, but for some invariants a single example suffices for all the questions.

For instance, the least $\pi$-weight (hence also $\pi$-character, density, and cellularity) of a ponderous compact space (hence also of a countably compact space that is not sequentially compact, etc.) is $\omega$, realized by the ponderous Hausdorff space $\beta\omega$. Were it not for the convention that every cardinal invariant gets multiplied by $\omega$, we could even lower this figure to 1 for [countably] compact spaces that are not sequentially compact: just add a single point $-\infty$ to any such space $X$, and have the topology be all unions of $\{-\infty\}$ with open sets of $X$, including the empty set—and, of course, add the empty set itself.

We also have a definitive example for the least character of a nonisolated point: it is $\omega_1$, while in a separable example it is $\mathfrak{p}$. The first figure obviously cannot be lowered; nor can the second, by the comments following Problem 6. Both figures are realized by ponderous compact Hausdorff spaces. For the first, take a quotient space of the Stone-Čech remainder of the discrete space of cardinality $\omega_1$: identify the closed subspace of uniform ultrafilters to a single point. For the second, use a quotient space of $\beta\omega$: let $\mathcal{B}$ be a filterbase on $\omega$ as in our definition of $\mathfrak{p}$, let $F = \bigcap\{B^* : B \in \mathcal{B}\}$, and identify $F$ to a point. [As usual, $B^* = c\ell_{\beta\omega}(B) \setminus B$.]

In any infinite Hausdorff space, indeed in any space with with an infinite family of disjoint open sets, precaliber and caliber are at least $\omega_1$, while $RO(X)$, the number of regular open sets, is at least $\mathfrak{c}$. All three are realized by the ponderous $\beta\omega$. For more general spaces, we can lower all three to $\omega$ where the two sequential compactness questions for them are concerned, using the $\pi$-weight $= 1$ trick above. By making a small modification, we can preserve the $T_1$ property. Given a $T_1$ space $X$ that is not sequentially compact, let $S$ be an uncountable set disjoint from $X$ and let the topology on $X \cup S$ be

$$\{\emptyset\} \cup \{U \cup T : U \in \tau(X) \text{ and } T = S \setminus F \text{ where } F \in [S]^{<\omega}\}.$$

Then $X \cup S$ is not sequentially compact either, but it is [countably] compact whenever $X$ is, and is of caliber $\omega$ because every infinite collection of nonempty open sets has nonempty intersection. For precaliber and $RO$, it is enough for $S$ to be denumerable: every family of nonempty open sets is centered, and the only regular open sets are the empty set and the whole space $X \cup S$!

The other two questions promise to be more difficult for all three invariants:

**Problem 10.** Is there a ponderous space of countable precaliber? one that is compact?

Although precaliber = caliber for compact Hausdorff spaces, they are distinct in general, as the example of $\omega$ with the cofinite topology shows.

**Problem 11.** Is there a ponderous space of countable caliber? one that is compact?

**Problem 12.** Is there a ponderous space with countably many (or at least fewer than $\mathfrak{c}$) regular open sets? one that is compact?

In Section 3, we said all we know about the least cardinality of a ponderous space. As for compact ponderous spaces, the best result to date is Dow's construction [D4] of a ponderous compact Hausdorff space of cardinality $\leq 2^{\mathfrak{s}}$ assuming the cofinality of $([\mathfrak{s}]^\omega, \subset)$ equals $\mathfrak{s}$. This hypothesis is so "weak" that its negation implies that there is an inner model with a proper class of measurable cardinals. This makes it reasonable to conjecture that the least cardinality of a compact ponderous space is no greater than $2^{\mathfrak{s}}$, and even this might not be optimal in all models.

The weight of Dow's example is $\mathfrak{c}$, but we conjecture that $\mathfrak{s}$ is the least weight of a ponderous compact space. [By Theorem C, it cannot be less.] There is an old construction of a ponderous compact Hausdorff space of weight $\mathfrak{s}$ in a model where $\omega_1 = \mathfrak{s} < \mathfrak{c}$: see [vDF] or the summary in [Ha]. More recently, Dow and Fremlin [DF] have shown that there is a ponderous compact Hausdorff space of weight $\mathfrak{s} = \omega_1$ in any model obtained by adding random reals to a model of CH. We know of no improvements to be had in going to more general spaces. We have yet to develop a technology for building non-Hausdorff compact ponderous spaces, and are only beginning to develop one for ponderous non-Hausdorff spaces in general.

In [Ny] there is a ZFC construction of a ponderous, locally countable space $Y$. Clearly, every compact subset of $Y$ is finite, so it is KC by default. Local countability implies that its pseudocharacter and tightness are $\omega$. It is also scattered, so its hereditary $\pi$-character is also $\omega$. So $\omega$ is also the least value of these invariants for a countably compact space that is not sequentially compact. For compact spaces it is a different story, since every locally countable compact space is countable and hence sequentially compact.

**Problem 13.** Must a compact space be of uncountable pseudocharacter if it is **(a)** ponderous, or **(b)** not sequentially compact?

The answer to both parts is affirmative in any model where $\mathfrak{c} < \mathfrak{n}$, because of Theorem 5 and Gryzlov's theorem that every compact $T_1$ space of countable pseudocharacter is of cardinality $\leq \mathfrak{c}$ [G]. Of course, pseudocharacter is only defined for $T_1$ spaces.

Obviously, no counterexample for Problem 13 can be $T_2$. At present we do not know of any improvement in ZFC on what we said about character in Section 4, as far as the least pseudocharacter of a compact space that is not sequentially compact is concerned. Where compact ponderous spaces are concerned, we have nothing better than what we have said about weight just now. These comments are also true of hereditary $\pi$-character, and for it we do not even have consistency results.

We can, however, restrict ourselves to $T_1$ spaces where minimum hereditary $\pi$-character of compact spaces that are not sequentially compact is concerned, just as we have been able to do already with all the invariants considered earlier, and all four basic questions. The key is the relation $x \leq y$ mentioned in the introduction, the one equivalent to $x \in c\ell\{y\}$. Minimal closed sets (if any) in a space are obviously of the form $c\ell\{y\}$, and if the space is $T_0$, they are singletons. We use the word **floor** for the union of the minimal closed subsets (if any) of a space. By Zorn's lemma, every point in a compact space is above some point in the floor, and if the space is $T_0$ then the floor is $T_1$. We also have:

**Theorem 9.** *The floor of a compact space is compact.*

*Proof.* Since every point is above some point in the floor, every open cover of the floor is automatically a cover of the whole space, and so it has a finite subcover. □

**Theorem 10.** *A compact space is sequentially compact if, and only if, its floor is sequentially compact.*

*Proof.* Let $X$ be compact. If its floor is sequentially compact and $\langle x_n : n \in \omega \rangle$ is a sequence in $X$, let $y_n$ be any point on the floor below $x_n$. If there are only finitely many distinct points of the form $y_n$, then there is an infinite subsequence of $\langle x_n : n \in \omega \rangle$ above one of them, and it converges to this point; otherwise, we can select a subsequence of $\langle x_n : n \in \omega \rangle$ such that the correspondence $x_n \to y_n$ is one-to-one, and if $y_n \to y$ then $x_n \to y$ also.

Conversely, if $X$ is sequentially compact and $\langle y_n : n \in \omega \rangle$ is a sequence in the floor, let $x$ be a limit of a convergent subsequence; then the sequence also converges to any point in the floor below $x$. □

So, if there is a compact space of hereditary $\pi$-character $\leq \kappa$ that is not sequentially compact, then there is a $T_0$ example as we saw in the introduction, and its floor is a $T_1$ example. The same reasoning applies to any hereditary cardinal invariant, and this includes spread, tightness, hereditary density, hereditary Lindelöf degree, and hereditary $\pi$-weight.

The least hereditary $\pi$-weight of a countably compact (wolog $T_1$) space which is not sequentially compact is $\leq \mathfrak{h}$, because $\mathfrak{h}$ is the least value of this invariant in Example 3, but we do not know whether this can be improved. Where the other three basic questions are concerned, we have not been able to do better than we did with weight.

We are slightly better off where tightness is concerned. Fedorchuk's ponderous hereditarily separable compact Hausdorff space constructed using $\diamondsuit$ [F] shows that it is consistent that it be $\omega$ for all four basic questions. On the other hand, the PFA implies that every compact Hausdorff space of countable tightness is sequential [B], and we have the following variation on the Moore-Mrówka problem which this PFA result settled:

**Problem 14.** Is there a (wolog $T_1$) compact space of countable tightness that is **(a)** ponderous, or at least **(b)** not sequentially compact?

The Moore-Mrówka problem asked whether there is a compact Hausdorff space of countable tightness which is not sequential, but we can modify the example used in proving (4) implies (1) to get a compact space of countable tightness that is not sequential: simply replace the $\omega_1$-tower by any complete tower. Also, the one-point compactification of the locally countable ponderous space $Y$ of [Ny] is a compact space of countable tightness that is not sequential, in which convergent sequences have unique limits. It is, however, sequentially compact, because every compact subset of $Y$ is finite and so every infinite 1-1 sequence converges to the extra point.

Fedorchuk's example also shows it is consistent for there to be a ponderous compact space of hereditary density and hence of spread $\omega$. For these two invariants we cannot do better in ZFC than $\mathfrak{s}$ [*resp.* $\mathfrak{h}$] for [countably] compact spaces that are not sequentially compact; these are due to both invariants being bounded above by net weight. The same is true of hereditary Lindelöf degree, which is uncountable in any model ZFC. For ponderous spaces we have nothing better than the results on $\mu_1$ in Section 3. For compact ponderous spaces, all we can say is that all the invariants in this paragraph, including net weight, are bounded above by weight and cardinality, and that their net weight and hereditary Lindelöf degree are bounded below by what we have established for compact spaces that are not sequentially compact.

The reasoning that gave us Theorem 10 also enables us to make short work of a slight weakening of the concept of ponderousness. Call a space **almost ponderous** if it is countably compact and has no convergent 1-1 sequences. This is clearly equivalent to being ponderous for a $T_1$ space, while in more general spaces it implies that each point is in the closure of only finitely many singletons. In particular, there cannot be a descending sequence of points of order type $\omega+1$ wrt the relation $x \leq y$ iff $x \in c\ell\{y\}$. But in a countably compact space, this is equivalent to every descending sequence being finite, and so every point is above some point in the floor, while every point in the floor has only finitely many points above it. It follows that a countably compact $T_0$ space is almost ponderous iff its floor (which is $T_1$) is ponderous, and that the least value of a cardinal invariant for an almost ponderous space is the same as for a ponderous space.

Finally, we look at $o(X) = |\tau(X)|$, the number of open sets. There is still a range of uncertainty for all four questions. The Stone-Čech compactification of $\omega$ is of little help here, since it has $2^{\mathfrak{c}}$ open sets. From what we have seen of $RO(X)$, it is clear that $\mathfrak{c} \leq o(X)$ for every Hausdorff space that is not sequentially compact, and the same is true of countably compact spaces that are not sequentially compact. The $T_1$ case is easy, even without assuming countable compactness:

**Theorem 11.** *Every $T_1$ space is either sequentially compact or contains an infinite discrete subspace.*

*Proof.* Suppose $X$ is $T_1$ and not sequentially compact, and let $\langle x_n : n \in \omega\rangle$ be a 1-1 sequence without a convergent subsequence. Let $p_0$ be any point of $X$ and let $U_0$ be an open nbhd of $p_0$ that omits infinitely many $x_n$. If $p_k$ and $U_k$ have been defined, let $p_{k+1}$ be outside $\bigcup_{i=0}^{k} U_i$ and let $U_{k+1}$ be an open nbhd of $p_{k+1}$ that omits $\{p_0, \dots, p_k\}$ along with infinitely many $x_n$ that are also omitted by $\bigcup_{i=0}^{k} U_i$. When the induction is complete, $\{p_n : n \in \omega\}$ is as desired. □

For arbitrary spaces, we need to assume countable compactness: the topology $\omega$ on $\omega$ shows we cannot drop $T_1$ from Theorem 11 [conciseness made possible by the von Neumann convention of identifying each ordinal with the set of smaller ordinals].

**Theorem 12.** *Every countably compact space is either sequentially compact or contains an infinite discrete subspace.*

*Proof.* Let $A = \{p_n : n \in \omega\}$ be as in the proof of Theorem 11, with the additional feature that each $p_k$ is one of the points $x_n$. There are two cases to consider.

*Case 1. Every infinite subset $B$ of $A$ contains a point $q$ and an infinite subset $C \subset B$ such that each point of $C$ has a neighborhood missing $q$.* In this case, let $A_0 = A$ and pick by induction a point $q_k$ of $A_k$ and an infinite $A_{k+1} \subset A$ such that each point of $A_{k+1}$ has a neighborhood missing $q_k$. Then $\{q_n : n \in \omega\}$ is discrete.

*Case 2. Otherwise.* Let $B$ be an infinite subset of $A$ such that for every point $q$ of $B$, all but finitely many points of $B$ are in the closure of $q$. Define by induction $\{q_n : n \in \omega\}$ such that each open set containing $q_n$ contains $\{q_k : k \leq n\}$. Let $x$ be a complete accumulation point of $\{q_n : n \in \omega\}$. Then every nbhd of $x$ contains the whole of $\{q_n : n \in \omega\}$, which thus converges to $x$, a contradiction. □

**Corollary 5.** *If $X$ is a countably compact space that is not sequentially compact, then $o(X) \geq \mathfrak{c}$.*

*Proof.* Let $D$ be a discrete subspace and, for each $d \in D$, let $U_\delta$ be an open nbhd of $d$ such that $U_d \cap D = \{d\}$. The sets of the form $\bigcup\{U_d : d \in E\}$ are open, and distinct for different $E \subset D$. □

Here is what we now know about the four basic questions for $o(X)$:

(1) For the least cardinality $\kappa_1$ of $o(X)$ for countably compact, non-sequentially compact $X$, we can say $\mathfrak{c} \leq \kappa_1 \leq 2^{\mathfrak{h}}$, because of Corollary 5 and Theorem 3. [Clearly $o(X) \leq 2^{nw(X)}$.]

(2) For the least cardinality $\kappa_2$ of $o(X)$ for compact, non-sequentially compact $X$, we can say $max\{\mathfrak{c}, \mathfrak{n}\} \leq \kappa_2 \leq 2^{\mathfrak{s}}$, because of Corollary 5, Theorem 5, Theorem 10 and Theorem C: clearly $|X| \leq o(X)$ for any $T_1$ space $X$.

(3) The least $o(X)$ for ponderous $X$ is $\leq 2^{\mu_1}$ which is consistently $< 2^{\mathfrak{c}}$ as noted in Section 3. We have no improvement on the lower bound in (1).

(4) The least $o(X)$ for compact ponderous $X$ is bounded above by 2 to the least weight, about which we have said all we can above. We have no improvement on the lower bound in (2).

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