Math 170- Graph Theory Notes

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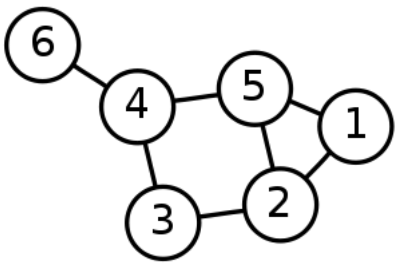
Notation: Let n be a positive integer. Denote $[n]$ to be the set $\{1, 2, \ldots, n\}$. So for example, $[3] = \{1, 2, 3\}$.

To quote Bud Brown, "Graph theory is a subject whose deceptive simplicity masks its vast applicability." Graph theory provides simple mathematical structures known as graphs to model the relations of various objects. The applications are numerous, including efficient storage of chemicals (graph coloring), optimal assignments (matchings), distribution networks (flows), efficient storage of data (tree-based data structures), and machine learning. In automata theory, we use directed graphs to provide a visual representation of our machines. Many elementary notions from graph theory, such as path-finding and walks, come up as a result. In complexity theory, many combinatorial optimization problems of interest are graph theoretic in nature. Therefore, it is important to discuss basic notions from graph theory. We begin with the basic definition of a graph.

Definition 1 (Simple Graph)**.** A simple graph is a two-tuple $G(V, E)$ where V is a set of vertices, and the edge set E is a set of 2-element subsets of V.

By convention, a simple graph is referred to as a *graph*, and an edge $\{i, j\}$ is written as ij. In simple graphs, $ij = ji$. Two vertices i, j are said to be *adjacent* if $ij \in E(G)$. Now let's consider an example of a graph.

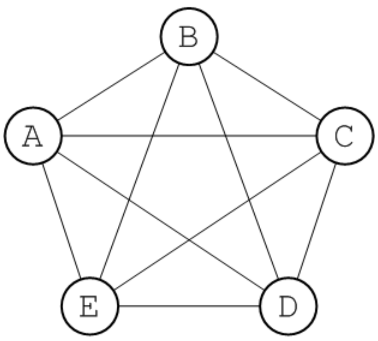
Example 1. Let $G(V, E)$ be the graph where $V = [6]$ and $E = \{12, 15, 23, 25, 34, 45, 56\}$. This graph is pictured below.



We now introduce several common classes of graphs.

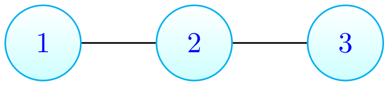
Definition 2 (Complete Graph)**.** The complete graph, denoted K_n, has the vertex set $V = [n]$ and edge set $E = \binom{V}{2}$. That is, K_n has all possible edges between vertices.

Example 2. The complete graph on five vertices K_5 is pictured below.



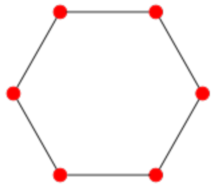
Definition 3 (Path Graph)**.** The path graph, dentoed P_n, has vertex set $V = [n]$ and the edge set $E = \{\{i, i+1\} : i \in [n-1]\}$.

Example 3. The path on three vertices P_3 is shown below.



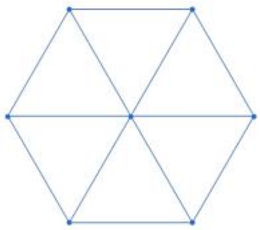
Definition 4 (Cycle Graph)**.** Let $n \geq 3$. The cycle graph, denoted C_n, has the vertex set $V = [n]$ and the edge set $E = \{\{i, i+1\} : i \in [n-1]\} \cup \{\{1, n\}\}$.

Example 4. Intuitively, C_n can be thought of as the regular n-gon. So C_3 is a triangle, C_4 is a quadrilateral, and C_5 is a pentagon. The graph C_6 is pictured below.



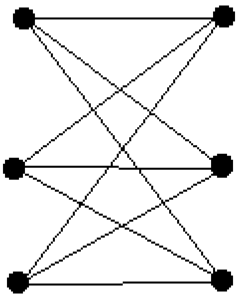
Definition 5 (Wheel Graph)**.** Let $n \geq 4$. The wheel graph, denoted W_n, is constructed by joining a vertex n to each vertex of C_{n-1}. So we take $C_{n-1} \dot\cup n$ and add the edges vn for each $v \in [n-1]$.

Example 5. The wheel graph on seven vertices W_7 is pictured below.



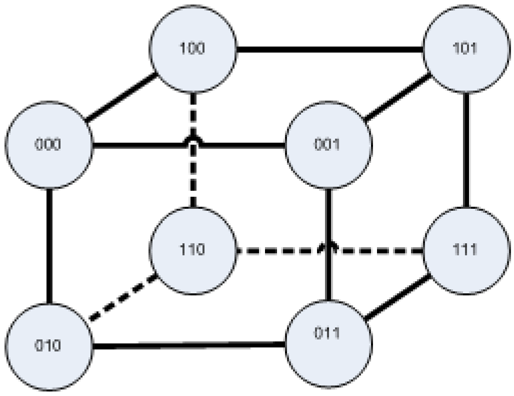
Definition 6 (Bipartite Graph)**.** A bipartite graph $G(V, E)$ has a vertex set V, which can be partitioned into two sets X and Y. That is, $V = X \cup Y$ and $X \cap Y = \emptyset$. The edge set of a bipartite graph E is a subset of $\{xy : x \in X, y \in Y\}$. That is, no two vertices in the same part of V are adjacent. So no two vertices in X are adjacent, and no two vertices in Y are adjacent.

Example 6. A common class of bipartite graphs include even-cycles C_{2n}. The complete bipartite graph is another common example. We denote the complete bipartite graph as $K_{m,n}$ which has vertex partitions $X \dot\cup Y$ where $|X| = m$ and $|Y| = n$. The edge set $E(K_{m,n}) = \{xy : x \in X, y \in Y\}$. The graph $K_{3,3}$ is pictured below.



Definition 7 (Hypercube)**.** The hypercube, denoted Q_n, has vertex set $V = \{0,1\}^n$. Two vertices are adjacent if the binary strings differ in precisely one component.

Example 7. The hypercube Q_2 is isomorphic to C_4 (isomorphism roughly means that two graphs are the same). The hypercube Q_3 is pictured below.



Q_3

Definition 8 (Connected Graph)**.** A graph $G(V,E)$ is said to be connected if for every $u, v \in V(G)$, there exists a $u - v$ path in G. A graph is said to be *disconnected* if it is not connected; and each connected subgraph is known as a *component*.

Example 8. So far, every graph presented has been connected. If we take two disjoint copies of any of the above graphs, their union forms a disconnected graph.

Definition 9 (Degree)**.** Let $G(V,E)$ be a graph and let $v \in V(G)$. The degree of v, dentoed $\deg(v)$ is the number of edges containing v. That is, $\deg(v) = |\{vx : vx \in E(G)\}|$.

Example 9. Each vertex in the Cycle graph C_n has degree 2. In Example 17, $\deg(6) = 1$ and $\deg(5) = 3$.

Theorem 0.1 (Handshake Lemma)**.** *Let $G(V,E)$ be a graph. We have $\sum_{v \in V(G)} deg(v) = 2|E(G)|$.*

Proof. The proof is by double counting. The term $\deg(v)$ counts the number of edges incident to v. Each edge has two endpoints v and x, for some other $x \in V(G)$. So the edge vx is double counted in both $\deg(v)$ and $\deg(x)$. Thus, $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$. □

Remark: The Handshake Lemma is a *necessary condition* for a graph to exist. That is, all graphs satisfy the Handshake Lemma. Consider the following: does there exist a graph on 11 vertices each having degree 5? By the Handshake Lemma, $11 \cdot 5 = 2|E(G)|$. However, 55 is not even, so no such graph exists. Note that the Handshake Lemma is not a *sufficient condition.* That is, there exist degree sequences such as $(3,3,1,1)$ satisfying the Handshake Lemma which are not realizable by any graph. Theorems such as Havel-Hakimi and Erdós-Gallai provide conditions that are both sufficient and necessary for a degree sequence to be realizable by some graph.

Next, the notion of a walk will be introduced. Walks on graphs come up frequently in automata theory. Intuitively, the sequence of transitions in an automaton is analogous to a walk on a graph. Additionally, algorithms like the State Reduction procedure and Brzozowski Algebraic Method that convert finite state automata to regular expressions are based on the idea of a walk on a graph.

Definition 10 (Walk)**.** Let $G(V,E)$ be a graph. A walk of length n is a sequence $(v_i)_{i=0}^{n}$ such that $v_i v_{i+1} \in E(G)$ for all $i \in \{0, \ldots, n-1\}$. If $v_0 = v_n$, the walk is said to be *closed.*

Let us develop some intuition for a walk. We start a given vertex v_0. Then we visit one of v_0's neighbors, which we call v_1. Next, we visit one of v_1's neighbors, which we call v_2. We continue this construction for the desired length of the walk. The key difference between a walk and a path is that a walk can repeat vertices, while all vertices in a path are distinct.

Example 10. Consider a walk on the hypercube Q_3. The sequence of vertices $(000, 100, 110, 111, 101)$ forms a walk, while $(000, 100, 110, 111, 101, 001, 000)$ is a closed walk. The sequence $(000, 111)$ is not a walk because 000 and 111 are not adjacent in Q_3.

We now define the adjacency matrix, which is useful for enumerating walks of a given length.

Definition 11 (Adjacency Matrix)**.** Let $G(V, E)$ be a graph. The adjacency matrix A is an $n \times n$ matrix where:

$$A_{ij} = \begin{cases} 1 & : ij \in E(G) \\ 0 & : ij \notin E(G) \end{cases} \tag{1}$$

Example 11. Consider the adjacency matrix for the graph K_5:

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \tag{2}$$

Theorem 0.2. *Let $G(V, E)$ be a graph, and let A be its adjacency matrix. For each $n \in \mathbb{Z}^+$, A^n_{ij} counts the number of walks of length n starting at vertex i and ending at vertex j.*

Proof. The proof is by induction on n. When $n = 1$, we have A. By definition $A_{ij} = 1$ iff $ij \in E(G)$. All walks of length 1 correspond to the edges incident to i, so the theorem holds true when $n = 1$. Now fix $k \geq 1$ and suppose that for each $m \in [k]$ that A^m_{ij} counts the number of $i{-}j$ walks of length m. The $k{+}1$ case will now be shown.

Consider $A^{k+1} = A^k \cdot A$ by associativity. By the inductive hypothesis, A^k_{ij} and A_{ij} count the number of $i - j$ walks of length k and 1 respectively. Observe that:

$$A^{k+1}_{ij} = \sum_{x=1}^{n} A^k_{ix} A_{xj}$$

So A^k_{ix} counts the number of ix walks of length k, and $A_{xj} = 1$ iff $xj \in E(G)$. Adding the edge xj to an $i - x$ walk of length k forms an $i - j$ walk of length $k + 1$. The result follows by induction. □

We prove one more theorem before concluding with the graph theory section. In order to prove this theorem, the following lemma (or helper theorem) will be introduced first.

Lemma 0.1. *Let $G(V, E)$ be a graph. Every closed walk of odd length in G contains an odd-cycle.*

Proof. The proof is by induction on the length of the walk. When we have a single vertex, this is trivially a cycle. Now suppose the that any closed walk of odd length up to $2k + 1$ has an odd-cycle. We prove true for walks of length $2k + 3$. Let $(v_i)_{i=0}^{2k+3}$ be a walk closed of odd length. If $v_0 = v_{2k+3}$, are the only repeated vertices, then the walk itself is an odd cycle and we are done. Otherwise, suppose $v_i = v_j$ for some $0 \leq i < j \leq 2k + 3$. If the walk $(v_t)_{t=i}^{k}$ is odd, then there exists an odd cycle by the inductive hypothesis. Otherwise, the walk $W = (v_0, \ldots, v_i, v_{j+1}, \ldots, v_{2k+3})$ is of odd length at most $2k + 1$. So by the IH, W has an odd cycle. So the lemma holds by induction. □

We now characterize bipartite graphs.

Theorem 0.3. *A graph $G(V, E)$ is bipartite if and only if it contains no cycles of odd length.*

Proof. Suppose first that G is bipartite with parts X and Y. Now consider a walk of length n. As no vertices in a fixed part are adjacent, only walks of even lengths can end back in the same part as the staring vertex. A cycle is a walk where all vertices are distinct, save for v_0 and v_n which are the same. Therefore, no cycle of odd length exists in G.

Conversely, suppose G has no cycles of odd length. We construct a bipartition of $V(G)$. Without loss of generality, suppose G is connected. For if G is not connected, we apply the same construction to each connected component. Fix the vertex v. Let $X = \{u \in V(G) : d(u, v) \text{ is even }\}$, where $d(u, v)$ denotes the distance or length of the shortest uv path. Let $Y = \{u \in V(G) : d(u, v) \text{ is odd }\}$. Clearly, $X \cap Y = \emptyset$. So it suffices to show no vertices within X are adjacent, and no vertices within Y are adjacent. Fix $v \in X$ and suppose to the contrary that two vertices in $y_1, y_2 \in Y$ are adjacent. Then there exists a closed walk of odd length $(v, \ldots, y_1, y_2, \ldots v)$. By Lemma 1.1, G must contain an odd-cycle, a contradiction. By similar argument, no vertices in X can be adjacent. So G is bipartite with bipartition $X \dot{\cup} Y$. $\square$